ELEMENTARY ANALYSIS

SMITH AND GRANVILLE

MATHEMATICAL TEXTS

FOR COLLEGES

EDITED BY

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PREFACE

The text of this volume is, to a considerable extent, identical with portions of corresponding chapters in Smith and Gale's "Elements of Analytic Geometry" and Granville's "Elements of the Differential and Integral Calculus." The new material is contained in the chapters on Curve Plotting (Chapter V) and Functions and Graphs (Chapter VI). At the same time, the parts which have appeared in previous books of the series have been thoroughly revised and, to a considerable extent, rewritten, to the end that the aim of the authors might be accomplished, — namely, to prepare a simple and direct exposition of those portions of mathematics beyond Trigonometry which are of importance to students of natural science. In this connection attention may be called to the intentional avoidance of anticipating difficulties, — a feature which is not common in textbooks. To particularize, processes which are natural are introduced without explanation, and exact definition is not given until the student is familiar by practice with the matter in hand. Again, in the derivation of certain formulas in the Differential Calculus the evaluation of particular limits is not undertaken until the student sees that this work *must* be done before the problem can be solved.

In many instances, when deemed wise, a general discussion is introduced by concrete examples. This feature, so common in school texts, is strangely absent from books intended for use in colleges and technical schools. Interest in the subject is usually aroused in this way, and it is the hope of the authors that this stimulus may not be lacking when the volume is studied.

New Haven, Connecticut

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ELEMENTARY ANALYSIS

CHAPTER I

FORMULAS FOR REFERENCE

1. Occasion will arise in later chapters to make use of the following formulas and theorems proved in geometry, algebra, and trigonometry.

1. Circumference of circle $= 2\pi r$.*
2. Area of circle $= \pi r^2$.
3. Volume of prism $= Ba$.
4. Volume of pyramid $= \frac{1}{3} Ba$.
5. Volume of right circular cylinder $= \pi r^2 a$.
6. Lateral surface of right circular cylinder $= 2\pi r a$.
7. Total surface of right circular cylinder $= 2\pi r(r + a)$.
8. Volume of right circular cone $= \frac{1}{3}\pi r^2 a$.
9. Lateral surface of right circular cone $= \pi r s$.
10. Total surface of right circular cone $= \pi r(r + s)$.
11. Volume of sphere $= \frac{4}{3}\pi r^3$.
12. Surface of sphere $= 4\pi r^2$.
13. In a geometrical series,

$$l = ar^{n-1};\ s = \frac{rl - a}{r - 1} = \frac{a(r^n - 1)}{r - 1},$$

$a =$ first term, $r =$ common ratio, $l = n$th term, $s =$ sum of n terms.

14. $\log ab = \log a + \log b$.
15. $\log \frac{a}{b} = \log a - \log b$.
16. $\log a^n = n \log a$.
17. $\log \sqrt[n]{a} = \frac{1}{n} \log a$.
18. $\log 1 = 0$.
19. $\log_a a = 1$.
20. $\log \frac{1}{a} = -\log a$.

* In formulas 1–12, r denotes radius, a altitude, B area of base, and s slant height.

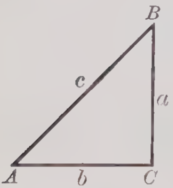
Functions of an angle in a right triangle. In any right triangle one of whose acute angles is A, the functions of A are defined as follows:

21. $$\sin A = \frac{\text{opposite side}}{\text{hypotenuse}}, \qquad \csc A = \frac{\text{hypotenuse}}{\text{opposite side}},$$

$$\cos A = \frac{\text{adjacent side}}{\text{hypotenuse}}, \qquad \sec A = \frac{\text{hypotenuse}}{\text{adjacent side}},$$

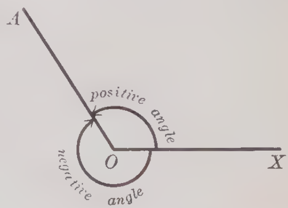
$$\tan A = \frac{\text{opposite side}}{\text{adjacent side}}, \qquad \cot A = \frac{\text{adjacent side}}{\text{opposite side}}.$$

From the above the theorem is easily derived:



22. In a right triangle a side is equal to the product of the hypotenuse and the sine of the angle opposite to that side, or of the hypotenuse and the cosine of the angle adjacent to that side.

Angles in general. In Trigonometry an angle XOA is considered as generated by the line OA rotating from an initial position OX. The angle is positive when OA rotates from OX *counter-clockwise*, and negative when the direction of rotation of OA is *clockwise*.



The fixed line OX is called the *initial line*, the line OA the *terminal line.*

Measurement of angles. There are two important methods of measuring angular magnitude, that is, there are two unit angles.

Degree measure. The unit angle is $\frac{1}{360}$ of a complete revolution, and is called a *degree.*

Circular measure. The unit angle is an angle whose subtending arc is equal to the radius of that arc, and is called a *radian.*

The fundamental relation between the unit angles is given by the equation

23. $$180 \textit{ degrees} = \pi \textit{ radians } (\pi = 3.14159 \cdots).$$

Or also, by solving this,

24. $$1 \text{ degree} = \frac{\pi}{180} = .0174 \cdots \text{ radians}.$$

25. $$1 \text{ radian} = \frac{180}{\pi} = 57.29 \cdots \text{ degrees}.$$

These equations enable us to change from one measurement to another. In the higher mathematics circular measure is always used, and will be adopted in this book.

The generating line is conceived of as rotating around O through as many revolutions as we choose. Hence the important result:

Any real number is the circular measure of some angle, and conversely, any angle is measured by a real number.

26. $\cot x = \dfrac{1}{\tan x}$; $\sec x = \dfrac{1}{\cos x}$; $\csc x = \dfrac{1}{\sin x}$.

27. $\tan x = \dfrac{\sin x}{\cos x}$; $\cot x = \dfrac{\cos x}{\sin x}$.

28. $\sin^2 x + \cos^2 x = 1$; $1 + \tan^2 x = \sec^2 x$; $1 + \cot^2 x = \csc^2 x$.

29. $\sin(-x) = -\sin x$; $\csc(-x) = -\csc x$;
$\cos(-x) = \cos x$; $\sec(-x) = \sec x$;
$\tan(-x) = -\tan x$; $\cot(-x) = -\cot x$.

30. $\sin(\pi - x) = \sin x$; $\sin(\pi + x) = -\sin x$;
$\cos(\pi - x) = -\cos x$; $\cos(\pi + x) = -\cos x$;
$\tan(\pi - x) = -\tan x$; $\tan(\pi + x) = \tan x$.

31. $\sin\left(\dfrac{\pi}{2} - x\right) = \cos x$; $\sin\left(\dfrac{\pi}{2} + x\right) = \cos x$;
$\cos\left(\dfrac{\pi}{2} - x\right) = \sin x$; $\cos\left(\dfrac{\pi}{2} + x\right) = -\sin x$;
$\tan\left(\dfrac{\pi}{2} - x\right) = \cot x$; $\tan\left(\dfrac{\pi}{2} + x\right) = -\cot x$.

32. $\sin(2\pi - x) = \sin(-x) = -\sin x$, etc.

33. $\sin(x + y) = \sin x \cos y + \cos x \sin y$.

34. $\sin(x - y) = \sin x \cos y - \cos x \sin y$.

35. $\cos(x + y) = \cos x \cos y - \sin x \sin y$.

36. $\cos(x - y) = \cos x \cos y + \sin x \sin y$.

37. $\tan(x + y) = \dfrac{\tan x + \tan y}{1 - \tan x \tan y}$. 38. $\tan(x - y) = \dfrac{\tan x - \tan y}{1 + \tan x \tan y}$.

39. $\sin 2x = 2 \sin x \cos x$; $\cos 2x = \cos^2 x - \sin^2 x$; $\tan 2x = \dfrac{2 \tan x}{1 - \tan^2 x}$.

40. $\sin\frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{2}}$; $\cos\frac{x}{2} = \pm\sqrt{\frac{1+\cos x}{2}}$; $\tan\frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{1+\cos x}}$.

41. $\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x$; $\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x$.

42. $\sin A - \sin B = 2\cos\frac{1}{2}(A+B)\sin\frac{1}{2}(A-B)$.

43. $\cos A - \cos B = -2\sin\frac{1}{2}(A+B)\sin\frac{1}{2}(A-B)$.

44. *Theorem. Law of cosines.* In any triangle the square of a side equals the sum of the squares of the two other sides diminished by twice the product of those sides by the cosine of their included angle;

that is, $$a^2 = b^2 + c^2 - 2bc\cos A.$$

45. *Theorem. Area of a triangle.* The area of any triangle equals one half the product of two sides by the sine of their included angle;

that is, $$area = \tfrac{1}{2}ab\sin C = \tfrac{1}{2}bc\sin A = \tfrac{1}{2}ca\sin B.$$

2. Three-place table of common logarithms of numbers.

N	0	1	2	3	4	5	6	7	8	9
1	000	041	079	114	146	176	204	230	255	279
2	301	322	342	362	380	398	415	431	447	462
3	477	491	505	518	532	544	556	568	580	591
4	602	613	623	634	643	653	663	672	681	690
5	699	708	716	724	732	740	748	756	763	771
6	778	785	792	799	806	813	820	826	832	839
7	845	851	857	863	869	875	881	886	892	898
8	903	908	914	919	924	929	934	939	944	949
9	954	959	964	968	973	978	982	987	991	996
10	000	004	009	013	017	021	025	029	033	037
11	041	045	049	053	057	061	064	068	072	076
12	079	083	086	090	093	097	100	104	107	111
13	114	117	121	124	127	130	134	137	140	143
14	146	149	152	155	158	161	164	167	170	173
15	176	179	182	185	188	190	193	196	199	201
16	204	207	210	212	215	218	220	223	225	228
17	230	233	236	238	241	243	246	248	250	253
18	255	258	260	262	265	267	270	272	274	276
19	279	281	283	286	288	290	292	294	297	299

3. Logarithms of trigonometric functions.

Angle in Radians	Angle in Degrees	log sin	log cos	log tan	log cot		
.000	0°	....	0.000	....	....	90°	1.571
.017	1°	8.242	9.999	8.242	1.758	89°	1.553
.035	2°	8.543	9.999	8.543	1.457	88°	1.536
.052	3°	8.719	9.999	8.719	1.281	87°	1.518
.070	4°	8.844	9.999	8.845	1.155	86°	1.501
.087	5°	8.940	9.998	8.942	1.058	85°	1.484
.174	10°	9.240	9.993	9.246	0.754	80°	1.396
.262	15°	9.413	9.985	9.428	0.572	75°	1.309
.349	20°	9.534	9.973	9.561	0.439	70°	1.222
.436	25°	9.626	9.957	9.669	0.331	65°	1.134
.524	30°	9.699	9.938	9.761	0.239	60°	1.047
.611	35°	9.759	9.913	9.845	0.165	55°	0.960
.698	40°	9.808	9.884	9.924	0.086	50°	0.873
.785	45°	9.850	9.850	0.000	0.000	45°	0.785
		log cos	log sin	log cot	log tan	Angle in Degrees	Angle in Radians

4. Natural values of trigonometric functions.

Angle in Radians	Angle in Degrees	sin	cos	tan	cot		
.000	0°	.000	1.000	.000	∞	90°	1.571
.017	1°	.017	.999	.017	57.29	89°	1.553
.035	2°	.035	.999	.035	28.64	88°	1.536
.052	3°	.052	.999	.052	19.08	87°	1.518
.070	4°	.070	.998	.070	14.30	86°	1.501
.087	5°	.087	.996	.088	11.43	85°	1.484
.174	10°	.174	.985	.176	5.67	80°	1.396
.262	15°	.259	.966	.268	3.73	75°	1.309
.349	20°	.342	.940	.364	2.75	70°	1.222
.436	25°	.423	.906	.466	2.14	65°	1.134
.524	30°	.500	.866	.577	1.73	60°	1.047
.611	35°	.574	.819	.700	1.43	55°	.960
.698	40°	.643	.766	.839	1.19	50°	.873
.785	45°	.707	.707	1.000	1.00	45°	.785
		cos	sin	cot	tan	Angle in Degrees	Angle in Radians

5. Special angles. Natural values.

Angle in Radians	Angle in Degrees	sin	cos	tan	cot	sec	csc
0	0°	0	1	0	∞	1	∞
$\frac{\pi}{2}$	90°	1	0	∞	0	∞	1
π	180°	0	-1	0	∞	-1	∞
$\frac{3\pi}{2}$	270°	-1	0	∞	0	∞	-1
2π	360°	0	1	0	∞	1	∞

Angle in Radians	Angle in Degrees	sin	cos	tan	cot	sec	csc
0	0°	0	1	0	∞	1	∞
$\frac{\pi}{6}$	30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	$\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	2
$\frac{\pi}{4}$	45°	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	1	$\sqrt{2}$	$\sqrt{2}$
$\frac{\pi}{3}$	60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{\sqrt{3}}{3}$	2	$\frac{2\sqrt{3}}{3}$
$\frac{\pi}{2}$	90°	1	0	∞	0	∞	1

The student should understand that the numerical values of the functions given in the second of the above tables are to be used when *formal* exactness only is required; that is, when it is sufficient to leave the results in radical form. *Explicit* numerical values are given in Art. 4. The same remarks apply to the values of the angle in radians.

6. Rules for signs.

Quadrant	sin	cos	tan	cot	sec	csc
First	+	+	+	+	+	+
Second	+	−	−	−	−	+
Third	−	−	+	+	−	−
Fourth	−	+	−	−	+	−

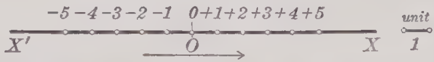
7. Greek alphabet.

Letters	Names	Letters	Names	Letters	Names
Α α	Alpha	Ι ι	Iota	Ρ ρ	Rho
Β β	Beta	Κ κ	Kappa	Σ σ ς	Sigma
Γ γ	Gamma	Λ λ	Lambda	Τ τ	Tau
Δ δ	Delta	Μ μ	Mu	Υ υ	Upsilon
Ε ε	Epsilon	Ν ν	Nu	Φ φ	Phi
Ζ ζ	Zeta	Ξ ξ	Xi	Χ χ	Chi
Η η	Eta	Ο ο	Omicron	Ψ ψ	Psi
Θ θ	Theta	Π π	Pi	Ω ω	Omega

CHAPTER II

CARTESIAN COÖRDINATES

8. Directed line. Let $X'X$ be an indefinite straight line, and let a point O, which we shall call the **origin**, be chosen upon it. Let a unit of length be adopted, and assume that lengths measured from O to the right are *positive*, and to the left *negative*.



Then any *real* number, if taken as the measure of the length of a line OP, will determine a point P on the line. Conversely, to each point P on the line will correspond a real number; namely, the measure of the length OP, with a positive or negative sign according as P is to the right or left of the origin.

The direction established upon $X'X$ by passing from the origin to the points corresponding to the positive numbers is called the **positive direction** on the line. A *directed line* is a

straight line upon which an origin, a unit of length, and a positive direction have been assumed.

An arrowhead is usually placed upon a directed line to indicate the positive direction.

If A and B are any two points of a directed line such that

$$OA = a, \quad OB = b,$$

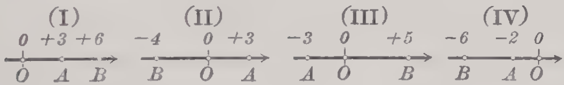
then the length of the segment AB is always given by $b - a$; that is, the length of AB is the difference of the numbers cor-

responding to B and A. This statement is evidently equivalent to the following definition:

For all positions of two points A and B on a directed line, the length AB is given by

$$(1)\qquad AB = OB - OA,$$

where O is the origin.



Illustrations.

In Fig.

I. $AB = OB - OA = 6 - 3 = +3$; $BA = OA - OB = 3 - 6 = -3$;

II. $AB = OB - OA = -4 - 3 = -7$; $BA = OA - OB = 3 - (-4) = +7$;

III. $AB = OB - OA = +5 - (-3) = +8$; $BA = OA - OB = -3 - 5 = -8$;

IV. $AB = OB - OA = -6 - (-2) = -4$; $BA = OA - OB = -2 - (-6) = +4$.

The following properties of lengths on a directed line are obvious:

(2) $AB = -BA$.

(3) AB is positive if the direction from A to B agrees with the positive direction on the line, and negative if in the contrary direction.

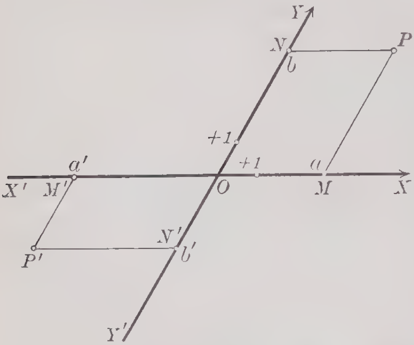
The phrase "distance between two points" should not be used if these points lie upon a directed line. Instead, we speak of the length AB, remembering that the lengths AB and BA are *not equal*, but that $AB = -BA$.

9. Cartesian* coördinates. Let $X'X$ and $Y'Y$ be two directed lines intersecting at O, and let P be any point in their plane. Draw lines through P parallel to $X'X$ and $Y'Y$ respectively. Then, if

$$OM = a, \quad ON = b,$$

* So called after René Descartes, 1596–1650, who first introduced the idea of coördinates into the study of Geometry.

the numbers a, b are called the *Cartesian coördinates* of P, a the **abscissa** and b the **ordinate**. The directed lines $X'X$ and $Y'Y$ are called the **axes** of coördinates, $X'X$ the **axis of abscissas,** $Y'Y$ the **axis of ordinates,** and their intersection O the *origin.*



The coördinates a, b of P are written (a, b), and the symbol $P(a, b)$ is to be read: "The point P, whose coördinates are a and b."

Any point P in the plane determines two numbers, the coördinates of P. Conversely, given two real numbers a' and b', then a point P' in the plane may always be constructed whose coördinates are (a', b'). For lay off $OM' = a'$, $ON' = b'$, and draw lines parallel to the axes through M' and N'. These lines intersect at $P'(a', b')$. Hence

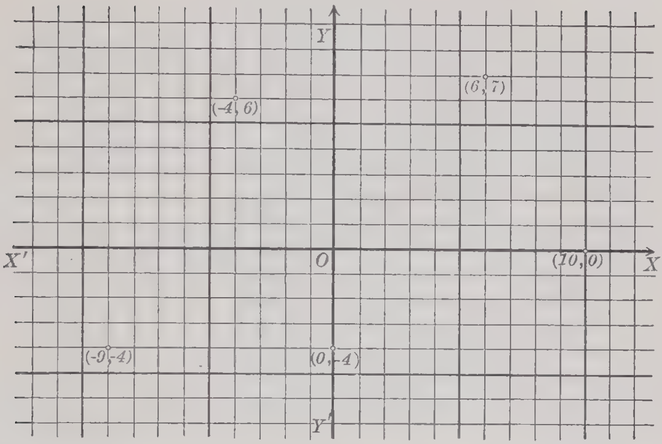
Every point determines a pair of real numbers, and conversely, a pair of real numbers determines a point.

The imaginary numbers of Algebra have no place in this representation, and for this reason elementary Analytic Geometry is concerned only with the real numbers of Algebra.

10. Rectangular coördinates. A rectangular system of coördinates is determined when the axes $X'X$ and $Y'Y$ are perpendicular to each other. This is the usual case, and will be assumed unless otherwise stated.

The work of plotting points in a rectangular system is much simplified by the use of *coördinate* or *plotting paper*, constructed by ruling off the plane into equal squares, the sides being parallel to the axes.

In the figure several points are plotted, the unit of length

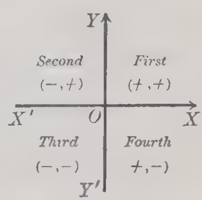


being assumed equal to one division on each axis. The method is simply this:

Count off from O along $X'X$ a number of divisions equal to the given abscissa, and then from the point so determined a number of divisions equal to the given ordinate, observing the

Rule *for signs:*

Abscissas are positive or negative according as they are laid off to the right or left of the origin. Ordinates are positive or negative according as they are laid off above or below the axis of x.



Rectangular axes divide the plane into four portions called *quadrants;* these are numbered as in the figure, in which the proper signs of the coördinates are also indicated.

As distinguished from rectangular coördinates, the term **oblique coördinates** is employed when the axes are not perpendicular, as in the figure, p. 10. The rule of signs given above applies to this case also.

In the following problems assume rectangular coördinates.

PROBLEMS

1. Plot accurately the points $(3, 2)$, $(3, -2)$, $(-4, 3)$, $(6, 0)$, $(-5, 0)$, $(0, 4)$.

2. What are the coördinates of the origin? *Ans.* $(0, 0)$.

3. In what quadrants do the following points lie if a and b are positive numbers: $(-a, b)$? $(-a, -b)$? $(b, -a)$? (a, b)?

4. To what quadrants is a point limited if its abscissa is positive? negative? its ordinate is positive? negative?

5. Plot the triangle whose vertices are $(2, -1)$, $(-2, 5)$, $(-8, -4)$.

6. Plot the triangle whose vertices are $(-2, 0)$, $(5\sqrt{3}-2, 5)$, $(-2, 10)$.

7. Plot the quadrilateral whose vertices are $(0, -2)$, $(4, 2)$, $(0, 6)$, $(-4, 2)$.

8. If a point moves parallel to the axis of x, which of its coördinates remains constant? if parallel to the axis of y?

9. Can a point move when its abscissa is zero? Where? Can it move when its ordinate is zero? Where? Can it move if both abscissa and ordinate are zero? Where will it be?

10. Where may a point be found if its abscissa is 2? if its ordinate is -3?

11. Where do all those points lie whose abscissas and ordinates are equal?

12. Two sides of a rectangle of lengths a and b coincide with the axes of x and y respectively. What are the coördinates of the vertices of the rectangle if it lies in the first quadrant? in the second quadrant? in the third quadrant? in the fourth quadrant?

13. Construct the quadrilateral whose vertices are $(-3, 6)$, $(-3, 0)$, $(3, 0)$, $(3, 6)$. What kind of a quadrilateral is it?

14. Show that (x, y) and $(x, -y)$ are symmetrical with respect to $X'X$; (x, y) and $(-x, y)$ with respect to $Y'Y$; and (x, y) and $(-x, -y)$ with respect to the origin.

15. A line joining two points is bisected at the origin. If the coördinates of one end are $(a, -b)$, what will be the coördinates of the other end?

16. Consider the bisectors of the angles between the coördinate axes. What is the relation between the abscissa and ordinate of any point of the bisector in the first and third quadrants? second and fourth quadrants?

17. A square whose side is $2a$ has its center at the origin. What will be the coördinates of its vertices if the sides are parallel to the axes? if the diagonals coincide with the axes?

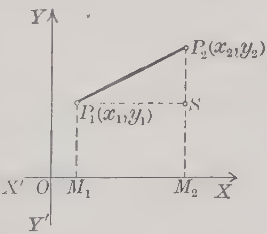
Ans. $(a, a), (a, -a), (-a, -a), (-a, a)$;
$(a\sqrt{2}, 0), (-a\sqrt{2}, 0), (0, a\sqrt{2}), (0, -a\sqrt{2})$.

18. An equilateral triangle whose side is a has its base on the axis of x and the opposite vertex above $X'X$. What are the vertices of the triangle if the center of the base is at the origin? if the lower left-hand vertex is at the origin?

Ans. $\left(\frac{a}{2}, 0\right), \left(-\frac{a}{2}, 0\right), \left(0, \frac{a\sqrt{3}}{2}\right)$;
$(0, 0), (a, 0), \left(\frac{a}{2}, \frac{a\sqrt{3}}{2}\right)$.

11. Lengths. Consider any two given points

$$P_1(x_1, y_1), P_2(x_2, y_2).$$



Then in the figure $OM_1 = x_1$, $OM_2 = x_2$, $M_1P_1 = y_1$, $M_2P_2 = y_2$.

We may now easily prove the important

Theorem. *The length l of the line joining two points $P_1(x_1, y_1)$, $P_2(x_2, y_2)$ is given by the formula*

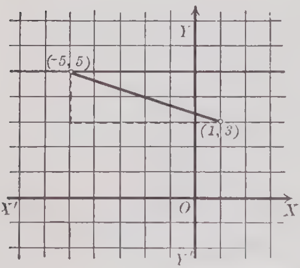
$$\text{(I)} \qquad l = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

Proof. Draw lines through P_1 and P_2 parallel to the axes to form the right triangle P_1SP_2.

Then
$$P_1S = OM_2 - OM_1 = x_2 - x_1,$$
$$SP_2 = M_2P_2 - M_1P_1 = y_2 - y_1,$$
$$P_1P_2 = \sqrt{\overline{SP_2}^2 + \overline{P_1S}^2};$$
and hence
$$l = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}. \qquad \text{Q.E.D}$$

The same method is used in deriving (I) for *any* positions of P_1 and P_2; namely, we construct a right triangle by drawing lines parallel to the axes through P_1 and P_2. The horizontal side of this triangle is equal to the difference of the abscissas of P_1 and P_2, while the vertical side is equal to the difference of the ordinates. The required length is then the square root of the sum of the *squares* of these sides, which gives (I). A number of different figures should be drawn to make the method clear.

EXAMPLE



Find the length of the line joining the points (1, 3) and $(-5, 5)$.

Solution. Call (1, 3) P_1, and $(-5, 5)$ P_2.

Then

$x_1 = 1$, $y_1 = 3$, and $x_2 = -5$, $y_2 = 5$;

and substituting in (I), we have

$$l = \sqrt{(1+5)^2 + (3-5)^2} = \sqrt{40} = 2\sqrt{10}.$$

It should be noticed that we are simply finding the hypotenuse of a right triangle whose sides are 6 and 2.

Remark. The fact that formula (I) is true for *all* positions of the points P_1 and P_2 is of fundamental importance. The application of this formula to any given problem is therefore simply a matter of direct substitution. In deriving such general formulas, it is most convenient to draw the figure so that the points lie in the first quadrant, or, in general, so that *all the quantities assumed as known shall be positive.*

PROBLEMS

1. Find the projections on the axes and the length of the lines joining the following points.

(*a*) $(-4, -4)$ and $(1, 3)$.

Ans. Projections 5, 7; length $= \sqrt{74}$.

(*b*) $(-\sqrt{2}, \sqrt{3})$ and $(\sqrt{3}, \sqrt{2})$.

Ans. Projections $\sqrt{3}+\sqrt{2}$, $\sqrt{2}-\sqrt{3}$; length $= \sqrt{10}$.

(*c*) $(0, 0)$ and $\left(\frac{a}{2}, \frac{a\sqrt{3}}{2}\right)$.

Ans. Projections $\frac{a}{2}, \frac{a}{2}\sqrt{3}$; length $= a$.

(*d*) $(a+b, c+a)$ and $(c+a, b+c)$.

Ans. Projections $c-b$, $b-a$; length $= \sqrt{(b-c)^2+(a-b)^2}$.

2. Find the projections of the sides of the following triangles upon the axes:

(*a*) $(0, 6), (1, 2), (3, -5)$.
(*b*) $(1, 0), (-1, -5), (-1, -8)$.
(*c*) $(a, b), (b, c), (c, d)$.
(*d*) $(a, -b), (b, -c), (c, -d)$.
(*e*) $(0, y), (-x, -y), (-x, 0)$.

3. Find the lengths of the sides of the triangles in problem 2.

4. Work out formula (I): (*a*) if $x_1 = x_2$; (*b*) if $y_1 = y_2$.

5. Find the lengths of the sides of the triangle whose vertices are $(4, 3)$, $(2, -2)$, $(-3, 5)$.

6. Show that the points $(1, 4)$, $(4, 1)$, $(5, 5)$ are the vertices of an isosceles triangle.

7. Show that the points $(2, 2)$, $(-2, -2)$, $(2\sqrt{3}, -2\sqrt{3})$ are the vertices of an equilateral triangle.

8. Show that $(3, 0)$, $(6, 4)$, $(-1, 3)$ are the vertices of a right triangle. What is its area?

9. Prove that $(-4, -2)$, $(2, 0)$, $(8, 6)$, $(2, 4)$ are the vertices of a parallelogram. Also find the lengths of the diagonals.

10. Show that $(11, 2)$, $(6, -10)$, $(-6, -5)$, $(-1, 7)$ are the vertices of a square. Find its area.

11. Show that the points $(1, 3)$, $(2, \sqrt{6})$, $(2, -\sqrt{6})$ are equidistant from the origin; that is, show that they lie on a circle with its center at the origin and its radius $\sqrt{10}$.

12. Show that the diagonals of any rectangle are equal.

13. Find the perimeter of the triangle whose vertices are (a, b), $(-a, b)$, $(-a, -b)$.

14. Find the perimeter of the polygon formed by joining the following points two by two in order:

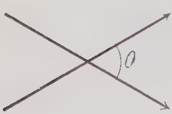
$$(6, 4), (4, -3), (0, -1), (-5, -4), (-2, 1).$$

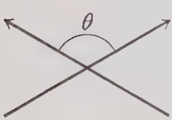
15. One end of a line whose length is 13 is the point $(-4, 8)$; the ordinate of the other end is 3. What is its abscissa? *Ans.* 8 or -16.

16. What equation must the coördinates of the point (x, y) satisfy if its distance from the point $(7, -2)$ is equal to 11?

17. What equation expresses algebraically the fact that the point (x, y) is equidistant from the points $(2, 3)$ and $(4, 5)$?

12. Inclination and slope. The angle between two intersecting directed lines is defined to be the angle made by their positive directions. In the figures the angle between the directed lines is the angle marked θ.





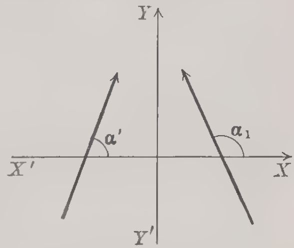
If the directed lines are parallel, then the angle between them is zero or π according as the positive directions agree or do not agree.

Evidently the angle between two directed lines may have any value from 0 to π inclusive. Reversing the direction of either di-

rected line changes θ to the supplement $\pi - \theta$. If both directions are reversed, the angle is unchanged.

When it is desired to assign a positive direction to a line intersecting $X'X$, we shall always assume the *upward direction as positive.*

The *inclination* of a line is the angle between the axis of x and the line when the latter is given the upward direction.

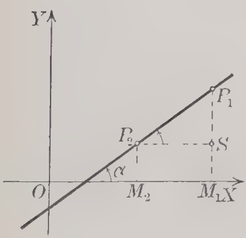


The *slope* of a line is the tangent of its inclination.

The inclination of a line will be denoted by α, α_1, α_2, α', etc.; its slope by m, m_1, m_2, m', etc., so that $m = \tan \alpha$, $m_1 = \tan \alpha_1$, etc.

The inclination may be any angle from 0 to π inclusive. The slope may be any real number, since the tangent of an angle in the first two quadrants may be any number positive or negative. The slope of a line parallel to $X'X$ is of course zero, since the inclination is 0 or π. For a line parallel to $Y'Y$ the slope is infinite.

Theorem. *The slope m of the line passing through two points $P_1(x_1, y_1)$, $P_2(x_2, y_2)$ is given by*



$$\text{(II)} \qquad m = \frac{y_1 - y_2}{x_1 - x_2}.$$

Proof. In the figure

$OM_1 = x_1$, $OM_2 = x_2$, $M_1P_1 = y_1$, $M_2P_2 = y_2$.

Draw P_2S parallel to OX. Then in the right triangle P_2SP_1, since angle $P_1P_2S = \alpha$, we have

$$(1) \qquad m = \tan \alpha = \frac{SP_1}{P_2S}.$$

But $$SP_1 = M_1P_1 - M_1S = M_1P_1 - M_2P_2 = y_1 - y_2;$$
$$P_2S = M_2M_1 = OM_1 - OM_2 = x_1 - x_2.$$

Substituting their values in (1), gives (II). Q.E.D.

The student should derive (II) when α is obtuse.*

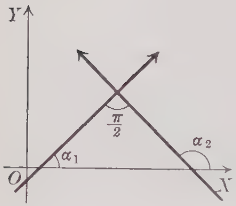
We next derive the conditions for parallel lines and for perpendicular lines, in terms of their slopes.

Theorem. *If two lines are parallel, their slopes are equal; if perpendicular, the slope of one is the negative reciprocal of the slope of the other, and conversely.*

Proof. Let α_1 and α_2 be the inclinations and m_1 and m_2 the slopes of the lines.

If the lines are parallel, $\alpha_1 = \alpha_2$. $\therefore m_1 = m_2$.

If the lines are perpendicular, as in the figure,



$$\alpha_2 = \frac{\pi}{2} + \alpha_1.$$

$$\therefore m_2 = \tan\alpha_2 = \tan\left(\frac{\pi}{2} + \alpha_1\right)$$

$$= -\cot\alpha_1 \qquad \text{(by 31, p. 3)}$$

$$= -\frac{1}{\tan\alpha_1}. \qquad \text{(by 26, p. 3)}$$

$$\therefore m_2 = -\frac{1}{m_1}.$$ Q.E.D.

The converse is proved by retracing the steps with the assumption, in the second part, that α_2 is greater than α_1.

The problem frequently arises: Given two points, to find the coördinates of their middle point. This is solved by means of

Theorem. *If $P(x, y)$ is the middle point of the line whose extremities are $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$, then*

* To construct a line passing through a given point P_1 whose slope is a positive fraction $\frac{a}{b}$, we mark a point S b units to the right of P_1 and a point P_2 a units above S, and draw P_1P_2. If the slope is a negative fraction, $-\frac{a}{b}$, then either S must lie to the left of P_1, or P_2 must lie below S.

(III) $$x = \tfrac{1}{2}(x_1 + x_2),\quad y = \tfrac{1}{2}(y_1 + y_2).$$

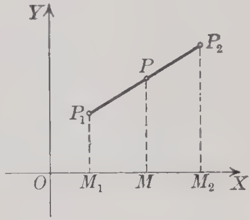
Proof. Project the points on OX. Then, by geometry, since $P_1P = PP_2$, then also

(1) $$M_1M = MM_2.$$

Now $OM_1 = x_1,\quad OM = x,\quad OM_2 = x_2.$

Hence $M_1M = x - x_1,\ MM_2 = x_2 - x.$

$\therefore x - x_1 = x_2 - x,\quad$ or $x = \tfrac{1}{2}(x_1 + x_2).$



Similarly, we may show that

$$y = \tfrac{1}{2}(y_1 + y_2).$$ Q.E.D.

PROBLEMS

1. Find the slope of the line joining (1, 3) and (2, 7).

Ans. 4.

2. Find the slope of the line joining (2, 7) and $(-4, -4)$.

Ans. $\frac{11}{6}$.

3. Find the slope of the line joining $(\sqrt{3}, \sqrt{2})$ and $(-\sqrt{2}, \sqrt{3})$. *Ans.* $2\sqrt{6} - 5$.

4. Find the slope of the line joining $(a + b, c + a)$, $(c + a, b + c)$. *Ans.* $\dfrac{b - a}{c - b}$.

5. Find the slopes of the sides of the triangle whose vertices are $(1, 1)$, $(-1, -1)$, $(\sqrt{3}, -\sqrt{3})$.

Ans. $1, \dfrac{1 + \sqrt{3}}{1 - \sqrt{3}}, \dfrac{1 - \sqrt{3}}{1 + \sqrt{3}}$.

6. Prove by means of slopes that $(-4, -2)$, $(2, 0)$, $(8, 6)$, $(2, 4)$ are the vertices of a parallelogram.

7. Prove by means of slopes that $(3, 0)$, $(6, 4)$, $(-1, 3)$ are the vertices of a right triangle.

8. Prove by means of slopes that $(0, -2)$, $(4, 2)$, $(0, 6)$, $(-4, 2)$ are the vertices of a rectangle, and hence, by (I), of a square.

9. Prove by means of their slopes that the diagonals of the square in problem 8 are perpendicular.

10. Prove by means of slopes that $(10, 0)$, $(5, 5)$, $(5, -5)$, $(-5, 5)$ are the vertices of a trapezoid.

11. Show that the line joining (a, b) and $(c, -d)$ is parallel to the line joining $(-a, -b)$ and $(-c, d)$.

12. Show that the line joining the origin to (a, b) is perpendicular to the line joining the origin to $(-b, a)$.

13. What is the inclination of a line parallel to $Y'Y$? perpendicular to $Y'Y$?

14. What is the slope of a line parallel to $Y'Y$? perpendicular to $Y'Y$?

15. What is the inclination of the line joining $(2, 2)$ and $(-2, -2)$?

Ans. $\frac{\pi}{4}$.

16. What is the inclination of the line joining $(-2, 0)$ and $(-5, 3)$?

Ans. $\frac{3\pi}{4}$.

17. What is the inclination of the line joining $(3, 0)$ and $(4, \sqrt{3})$?

Ans. $\frac{\pi}{3}$.

18. What is the inclination of the line joining $(3, 0)$ and $(2, \sqrt{3})$?

Ans. $\frac{2\pi}{3}$.

19. What is the inclination of the line joining $(0, -4)$ and $(-\sqrt{3}, -5)$?

Ans. $\frac{\pi}{6}$.

20. What is the inclination of the line joining $(0, 0)$ and $(-\sqrt{3}, 1)$?

Ans. $\frac{5\pi}{6}$.

21. Prove by means of slopes that $(2, 3)$, $(1, -3)$, $(3, 9)$ lie on the same straight line.

22. Prove that the points $(a, b + c)$, $(b, c + a)$, and $(c, a + b)$ lie on the same straight line.

23. Prove that $(1, 5)$ is on the line joining the points $(0, 2)$ and $(2, 8)$ and is equidistant from them.

24. Prove that the line joining $(3, -2)$ and $(5, 1)$ is perpendicular to the line joining $(10, 0)$ and $(13, -2)$.

25. Find the coördinates of the middle point of the line joining $(4, -6)$ and $(-2, -4)$. *Ans.* $(1, -5)$.

26. Find the coördinates of the middle point of the line joining $(a+b, c+d)$ and $(a-b, d-c)$. *Ans.* (a, d).

27. Find the middle points of the sides of the triangle whose vertices are $(2, 3)$, $(4, -5)$, and $(-3, -6)$; also find the lengths of the medians.

28. Prove that the middle point of the hypotenuse of a right triangle is equidistant from the three vertices.

29. Show that the diagonals of the parallelogram whose vertices are $(1, 2)$, $(-5, -3)$, $(7, -6)$, $(1, -11)$ bisect each other.

30. Prove that the diagonals of any parallelogram mutually bisect each other.

31. Show that the lines joining the middle points of the opposite sides of the quadrilateral whose vertices are $(6, 8)$, $(-4, 0)$, $(-2, -6)$, $(4, -4)$ bisect each other.

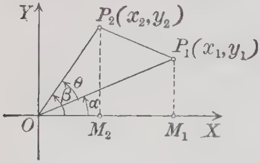
32. In the quadrilateral of problem 31 show by means of slopes that the lines joining the middle points of the adjacent sides form a parallelogram.

33. Show that in the trapezoid whose vertices are $(-8, 0)$, $(-4, -4)$, $(-4, 4)$, and $(4, -4)$ the length of the line joining the middle points of the non-parallel sides is equal to one half the sum of the lengths of the parallel sides. Also prove that it is parallel to the parallel sides.

13. Areas. In this section the problem of determining the area of any polygon, the coördinates of whose vertices are given, will be solved. We begin with

Theorem. *The area of a triangle whose vertices are the origin, $P_1(x_1, y_1)$, and $P_2(x_2, y_2)$ is given by the formula*

(IV) **Area of triangle $OP_1P_2 = \frac{1}{2}(x_1y_2 - x_2y_1)$.**



Proof. In the figure let

$$\alpha = \angle XOP_1,$$
$$\beta = \angle XOP_2,$$
$$\theta = \angle P_1OP_2.$$

(1) $$\therefore \theta = \beta - \alpha.$$

By 45, p. 4,

(2) $$\text{Area } \triangle OP_1P_2 = \tfrac{1}{2}\, OP_1 \cdot OP_2 \sin\theta$$
$$= \tfrac{1}{2}\, OP_1 \cdot OP_2 \sin(\beta - \alpha) \qquad \text{[by (1)]}$$
(3) $$= \tfrac{1}{2}\, OP_1 \cdot OP_2 (\sin\beta\cos\alpha - \cos\beta\sin\alpha).$$
(by 34, p. 3)

But in the figure

$$\sin\beta = \frac{M_2P_2}{OP_2} = \frac{y_2}{OP_2},\ \cos\beta = \frac{OM_2}{OP_2} = \frac{x_2}{OP_2},$$

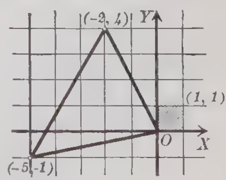
$$\sin\alpha = \frac{M_1P_1}{OP_1} = \frac{y_1}{OP_1},\ \cos\alpha = \frac{OM_1}{OP_1} = \frac{x_1}{OP_1}.$$

Substituting in (3) and reducing, we obtain

$$\text{Area } \triangle OP_1P_2 = \tfrac{1}{2}(x_1y_2 - x_2y_1). \qquad \text{Q.E.D.}$$

EXAMPLE

Find the area of the triangle whose vertices are the origin, $(-2, 4)$, and $(-5, -1)$.



Solution. Denote $(-2, 4)$ by P_1, $(-5, -1)$ by P_2. Then

$$x_1 = -2,\ y_1 = 4,\ x_2 = -5,\ y_2 = -1.$$

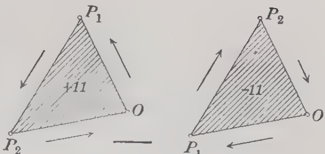
Substituting in (IV),

$$\text{Area} = \tfrac{1}{2}[-2 \cdot -1 - (-5) \cdot 4] = 11.$$

Then Area $= 11$ *unit squares.*

If, however, the formula (IV) is applied by denoting $(-2, 4)$ by P_2, and $(-5, -1)$ by P_1, the result will be -11.

The two figures are as follows:



(1) (2)

The cases of *positive* and *negative* area are distinguished by

Theorem. *Passing around the perimeter in the order of the vertices O, P_1, P_2,*

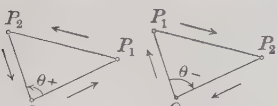
if the area is on the left, as in Fig. 1, then (IV) *gives a positive result;*

if the area is on the right, as in Fig. 2, then (IV) *gives a negative result.*

Proof. In the formula

$$\text{(4)} \qquad \text{Area } \triangle\, OP_1P_2 = \tfrac{1}{2}\, OP_1 \cdot OP_2 \sin\theta$$

the angle θ is measured *from OP_1 to OP_2 within the triangle.*



(1) (2)

Hence θ is positive when the area lies to the left in passing around the perimeter O, P_1, P_2, as in Fig. 1, since θ is then measured counter-clockwise (p. 2). But in Fig. 2, θ is measured clockwise. Hence θ is *negative* and $\sin\theta$ in (4) is also negative.

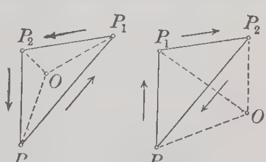
Q.E.D.

Formula (IV) is easily applied to any polygon by regarding its area as made up of triangles with the origin as a common vertex. Consider any triangle.

Theorem. *The area of a triangle whose vertices are $P_1(x_1, y_1)$, $P_2(x_2, y_2)$, $P_3(x_3, y_3)$ is given by*

$$\textbf{(V)} \quad \textbf{Area } \triangle\, \boldsymbol{P_1P_2P_3} = \tfrac{1}{2}(\boldsymbol{x_1y_2 - x_2y_1 + x_2y_3 - x_3y_2 + x_3y_1 - x_1y_3}).$$

This formula gives a positive or negative result according as the area lies to the left or right in passing around the perimeter in the order $P_1\, P_2\, P_3$.



(1) (2)

Proof. Two cases must be distinguished according as the origin is within or without the triangle.

Fig. 1, *origin within the triangle.* By inspection,

(5) $\quad$ Area $\triangle P_1P_2P_3 = \triangle OP_1P_2 + \triangle OP_2P_3 + \triangle OP_3P_1$,

since these areas all have the *same sign.*

Fig. 2, *origin without the triangle.* By inspection,

(6) $\quad$ Area $\triangle P_1P_2P_3 = \triangle OP_1P_2 + \triangle OP_2P_3 + \triangle OP_3P_1$,

since OP_1P_2, OP_3P_1 have the *same* sign, but OP_2P_3 the opposite sign, the *algebraic sum* giving the desired area.

By (IV), $\triangle OP_1P_2 = \frac{1}{2}(x_1y_2 - x_2y_1)$,

$$\triangle OP_2P_3 = \tfrac{1}{2}(x_2y_3 - x_3y_2),\ \triangle OP_3P_1 = \tfrac{1}{2}(x_3y_1 - x_1y_3).$$

Substituting in (5) and (6), we have (V).

Also in (5) the area is positive, in (6) negative. Q.E.D.

An easy way to apply (V) is given by the following

Rule *for finding the area of a triangle.*

First step. *Write down the vertices in two columns, abscissas in one, ordinates in the other, repeating the coördinates of the first vertex.*

$$\begin{array}{cc} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_1 & y_1 \end{array}$$

Second step. *Multiply each abscissa by the ordinate of the next row, and add results. This gives* $x_1y_2 + x_2y_3 + x_3y_1$.

Third step. *Multiply each ordinate by the abscissa of the next row, and add results. This gives* $y_1x_2 + y_2x_3 + y_3x_1$.

Fourth step. *Subtract the result of the third step from that of the second step, and divide by 2. This gives the required area,* namely, formula (V).

It is easy to show in the same manner that the rule applies to any polygon, if the following caution be observed in the first step:

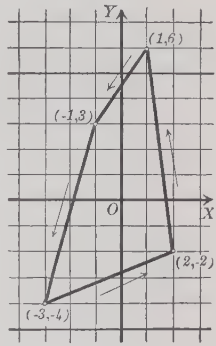
Write down the coördinates of the vertices in an order agreeing with that established by passing continuously around the perimeter, and repeat the coördinates of the first vertex.

EXAMPLE

Find the area of the quadrilateral whose vertices are (1, 6), (– 3, – 4), (2, – 2), (– 1, 3).

Solution. Plotting, we have the figure from which we choose the *order* of the vertices as indicated by the arrows. Following the rule:

1	6
– 1	3
– 3	– 4
2	– 2
1	6



First step. Write down the vertices in order.

Second step. Multiply each abscissa by the ordinate of the next row, and add. This gives

$$1 \times 3 + (-1 \times -4) + (-3 \times -2) + 2 \times 6 = 25.$$

Third step. Multiply each ordinate by the abscissa of the next row, and add. This gives

$$6 \times -1 + 3 \times -3 + (-4 \times 2) + (-2 \times 1) = -25.$$

Fourth step. Subtract the result of the third step from the result of the second step, and divide by 2.

$$\therefore \text{Area} = \frac{25 + 25}{2} = 25 \text{ unit squares.} \quad \textit{Ans.}$$

The result has the positive sign, since the area is on the *left*.

PROBLEMS

1. Find the area of the triangle whose vertices are (2, 3), (1, 5), (– 1, – 2). *Ans.* $\frac{11}{2}$.

2. Find the area of the triangle whose vertices are (2, 3), (4, – 5) (– 3, – 6). *Ans.* 29.

3. Find the area of the triangle whose vertices are (8, 3), (– 2, 3), (4, – 5). *Ans.* 40.

4. Find the area of the triangle whose vertices are (a, 0), (– a, 0), (0, b). *Ans.* ab.

5. Find the area of the triangle whose vertices are $(0, 0)$, (x_1, y_1), (x_2, y_2). *Ans.* $\dfrac{x_1y_2 - x_2y_1}{2}$.

6. Find the area of the triangle whose vertices are $(a, 1)$, $(0, b)$, $(c, 1)$. *Ans.* $\dfrac{(a-c)(b-1)}{2}$.

7. Find the area of the triangle whose vertices are (a, b), (b, a), $(c, -c)$. *Ans.* $\frac{1}{2}(a^2 - b^2)$.

8. Find the area of the triangle whose vertices are $(3, 0)$, $(0, 3\sqrt{3})$, $(6, 3\sqrt{3})$. *Ans.* $9\sqrt{3}$.

9. Prove that the area of the triangle whose vertices are the points $(2, 3)$, $(5, 4)$, $(-4, 1)$ is zero, and hence that these points all lie on the same straight line.

10. Prove that the area of the triangle whose vertices are the points $(a, b+c)$, $(b, c+a)$, $(c, a+b)$ is zero, and hence that these points all lie on the same straight line.

11. Prove that the area of the triangle whose vertices are the points $(a, c+a)$, $(-c, 0)$, $(-a, c-a)$ is zero, and hence that these points all lie on the same straight line.

12. Find the area of the quadrilateral whose vertices are $(-2, 3)$, $(-3, -4)$, $(5, -1)$, $(2, 2)$. *Ans.* 31.

13. Find the area of the pentagon whose vertices are $(1, 2)$, $(3, -1)$, $(6, -2)$, $(2, 5)$, $(4, 4)$. *Ans.* 18.

14. Find the area of the parallelogram whose vertices are $(10, 5)$, $(-2, 5)$, $(-5, -3)$, $(7, -3)$. *Ans.* 96.

15. Find the area of the quadrilateral whose vertices are $(0, 0)$, $(5, 0)$, $(9, 11)$, $(0, 3)$. *Ans.* 41.

16. Find the area of the quadrilateral whose vertices are $(7, 0)$, $(11, 9)$, $(0, 5)$, $(0, 0)$. *Ans.* 59.

17. Show that the area of the triangle whose vertices are $(4, 6)$, $(2, -4)$, $(-4, 2)$ is four times the area of the triangle formed by joining the middle points of the sides.

18. Show that the lines drawn from the vertices $(3, -8)$, $(-4, 6)$, $(7, 0)$ to the medial point of the triangle divide it into three triangles of equal area.

19. Given the quadrilateral whose vertices are $(0, 0)$, $(6, 8)$, $(10, -2)$, $(4, -4)$; show that the area of the quadrilateral formed by joining the middle points of its adjacent sides is equal to one half the area of the given quadrilateral.

CHAPTER III

CURVE AND EQUATION

14. Locus of a point satisfying a given condition. The curve* (or group of curves) passing through all points which satisfy a given condition, and through no other points, is called the **locus** of the point satisfying that condition.

For example, in Plane Geometry, the following results are proved:

The perpendicular bisector of the line joining two fixed points is the locus of all points equidistant from these points.

The bisectors of the adjacent angles formed by two lines are the locus of all points equidistant from these lines.

To solve any locus problem involves two things:

1. To draw the locus by constructing a sufficient number of points satisfying the given condition and therefore lying on the locus.

2. To discuss the nature of the locus, that is, to determine properties of the curve.

Analytic Geometry is peculiarly adapted to the solution of both parts of a locus problem.

15. Equation of the locus of a point satisfying a given condition. Let us take up the locus problem, making use of coördinates. We imagine the point $P(x, y)$ *moving* in such a manner that the given condition is fulfilled. Then the given condition will lead to an equation involving the variables x and y. The following example illustrates this.

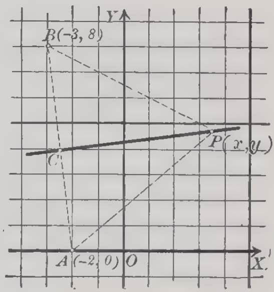
* The word "curve" will hereafter signify *any continuous line*, straight or curved.

EXAMPLE

The point $P(x, y)$ moves so that it is always equidistant from $A(-2, 0)$ and $B(-3, 8)$. Find the equation of the locus.

Solution. Let $P(x, y)$ be any point on the locus. Then by the given condition

(1) $PA = PB.$



But, by formula (I), p. 13,

$$PA = \sqrt{(x+2)^2 + (y-0)^2},$$

$$PB = \sqrt{(x+3)^2 + (y-8)^2}.$$

Substituting in (1),

(2) $$\sqrt{(x+2)^2+(y-0)^2} = \sqrt{(x+3)^2+(y-8)^2}.$$

Squaring and reducing,

(3) $$2x - 16y + 69 = 0.$$

In the equation (3), x and y are *variables* representing the coördinates of any point on the locus; that is, of any point on the perpendicular bisector of the line AB. This equation is called the equation of the locus; that is, it is the equation of the perpendicular bisector CP. It has two important and characteristic properties:

1. The coördinates of any point on the locus may be substituted for x and y in the equation (3), and the result will be true.

For let $P_1(x_1, y_1)$ be any point on the locus. Then $P_1A = P_1B$, by definition. Hence, by formula (I), p. 13,

(4) $$\sqrt{(x_1+2)^2 + y_1^2} = \sqrt{(x_1+3)^2 + (y_1-8)^2},$$

or, squaring and reducing,

(5) $$2x_1 - 16y_1 + 69 = 0.$$

But this equation is obtained by substituting x_1 and y_1 for x and y, respectively, in (3). Therefore x_1 and y_1 satisfy (3).

2. Conversely, every point whose coördinates satisfy (3) will lie upon the locus.

For if $P_1(x_1, y_1)$ is a point whose coördinates satisfy (3), then (5) is true, and hence also (4) holds. Q.E.D.

In particular, the coördinates of the middle point C of A and B, namely, $x = -2\frac{1}{2}$, $y = 4$ (III, p. 19), satisfy (3), since

$$2(-2\tfrac{1}{2}) - 16 \times 4 + 69 = 0.$$

This discussion leads to the definition:

The **equation of the locus** of a point satisfying a given condition is an equation in the variables x and y representing coördinates such that (1) the coördinates of every point on the locus will satisfy the equation; and (2) conversely, every point whose coördinates satisfy the equation will lie upon the locus.

This definition shows that the equation of the locus must be tested *in two ways* after derivation, as illustrated in the example of this section. The student should supply this test in the examples, p. 31, and problems, p. 32.

From the above definition follows at once the

Corollary. *A point lies upon a curve when and only when its coördinates satisfy the equation of the curve.*

16. First fundamental problem. *To find the equation of a curve which is defined as the locus of a point satisfying a given condition.*

The following rule will suffice for the solution of this problem in many cases:

Rule. First step. *Assume that $P(x, y)$ is any point satisfying the given condition and is therefore on the curve.*

Second step. *Write down the given condition.*

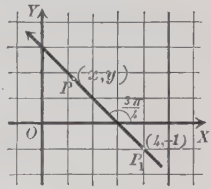
Third step. *Express the given condition in coördinates, and simplify the result. The final equation, containing x, y, and the given constants of the problem, will be the required equation.*

EXAMPLES

1. Find the equation of the straight line passing through $P_1(4, -1)$ and having an inclination of $\frac{3\pi}{4}$.

Solution. *First step.* Assume $P(x, y)$ any point on the line.

Second step. The given condition, since the inclination α is $\frac{3\pi}{4}$, may be written



(1) Slope of $P_1P = \tan\alpha = -1$.

Third step. From (II), p. 17,

(2) $$\text{Slope of } P_1P = \tan\alpha = \frac{y_1 - y_2}{x_1 - x_2} = \frac{y+1}{x-4}.$$

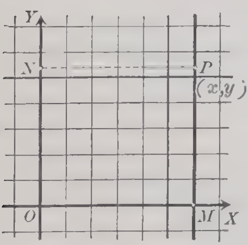
[By substituting (x, y) for (x_1, y_1), and $(4, -1)$ for (x_2, y_2).]

$\therefore$ from (1), $$\frac{y+1}{x-4} = -1,$$

or

(3) $$x + y - 3 = 0. \quad \textit{Ans.}$$

2. Find the equation of a straight line parallel to the axis of y and at a distance of 6 units to the right.



Solution. *First step.* Assume that $P(x, y)$ is any point on the line, and draw NP perpendicular to OY.

Second step. The given condition may be written

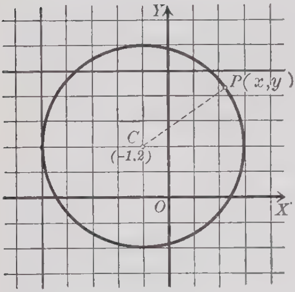
(4) $NP = 6$.

Third step. Since $NP = OM = x$, (4) becomes

(5) $$x = 6. \quad \textit{Ans.}$$

3. Find the equation of the locus of a point whose distance from $(-1, 2)$ is always equal to 4.

Solution. *First step.* Assume that $P(x, y)$ is any point on the locus.



Second step. Denoting $(-1, 2)$ by C, the given condition is

(6) $$PC = 4.$$

Third step. By formula (I), p. 13,

$$PC = \sqrt{(x+1)^2 + (y-2)^2}.$$

Substituting in (6),

$$\sqrt{(x+1)^2 + (y-2)^2} = 4.$$

Squaring and reducing,

(7) $$x^2 + y^2 + 2x - 4y - 11 = 0.$$

This is the required equation, namely, the equation of the circle whose center is $(-1, 2)$ and radius equal 4.

PROBLEMS

1. Find the equation of a line parallel to OY and

(*a*) at a distance of 4 units to the right.
(*b*) at a distance of 7 units to the left.
(*c*) at a distance of 2 units to the right of $(3, 2)$.
(*d*) at a distance of 5 units to the left of $(2, -2)$.

2. Find the equation of a line parallel to OX and

(*a*) at a distance of 3 units above OX.
(*b*) at a distance of 6 units below OX.
(*c*) at a distance of 7 units above $(-2, -3)$.
(*d*) at a distance of 5 units below $(4, -2)$.

3. What is the equation of XX'? of YY'?

4. Find the equation of a line parallel to the line $x = 4$ and 3 units to the right of it. Eight units to the left of it.

5. Find the equation of a line parallel to the line $y = -2$ and 4 units below it. Five units above it.

6. What is the equation of the locus of a point which moves always at a distance of 2 units from the axis of x? from the axis of y? from the line $x=-5$? from the line $y=4$?

7. What is the equation of the locus of a point which moves so as to be equidistant from the lines $x=5$ and $x=9$? equidistant from $y=3$ and $y=-7$?

8. What are the equations of the sides of the rectangle whose vertices are $(5, 2)$, $(5, 5)$, $(-2, 2)$, $(-2, 5)$?

In problems 9 and 10, P_1 is a given point on the required line, m is the slope of the line, and α its inclination.

9. What is the equation of a line if

(*a*) P_1 is $(0, 3)$ and $m=-3$? *Ans.* $3x+y-3=0$.
(*b*) P_1 is $(-4, -2)$ and $m=\frac{1}{3}$? *Ans.* $x-3y-2=0$.
(*c*) P_1 is $(-2, 3)$ and $m=\frac{\sqrt{2}}{2}$? *Ans.* $\sqrt{2}x-2y+6+2\sqrt{2}=0$.
(*d*) P_1 is $(0, 5)$ and $m=\frac{\sqrt{3}}{2}$? *Ans.* $\sqrt{3}x-2y+10=0$.
(*e*) P_1 is $(0, 0)$ and $m=-\frac{2}{3}$? *Ans.* $2x+3y=0$.
(*f*) P_1 is (a, b) and $m=0$? *Ans.* $y=b$.
(*g*) P_1 is $(-a, b)$ and $m=\infty$? *Ans.* $x=-a$.

10. What is the equation of a line if

(*a*) P_1 is $(2, 3)$ and $\alpha=45°$? *Ans.* $x-y+1=0$.
(*b*) P_1 is $(-1, 2)$ and $\alpha=45°$? *Ans.* $x-y+3=0$.
(*c*) P_1 is $(-a, -b)$ and $\alpha=45°$? *Ans.* $x-y=b-a$.
(*d*) P_1 is $(5, 2)$ and $\alpha=60°$? *Ans.* $\sqrt{3}x-y+2-5\sqrt{3}=0$.
(*e*) P_1 is $(0, -7)$ and $\alpha=60°$? *Ans.* $\sqrt{3}x-y-7=0$.
(*f*) P_1 is $(-4, 5)$ and $\alpha=0°$? *Ans.* $y=5$.
(*g*) P_1 is $(2, -3)$ and $\alpha=90°$? *Ans.* $x=2$.
(*h*) P_1 is $(3, -3\sqrt{3})$ and $\alpha=120°$? *Ans.* $\sqrt{3}x+y=0$.
(*i*) P_1 is $(0, 3)$ and $\alpha=150°$? *Ans.* $\sqrt{3}x+3y-9=0$.
(*j*) P_1 is (a, b) and $\alpha=135°$? *Ans.* $x+y=a+b$.

11. Find the equation of the circle with

(*a*) center at (3, 2) and radius $= 4$.

Ans. $x^2 + y^2 - 6x - 4y - 3 = 0$.

(*b*) center at (12, -5) and $r = 13$.

Ans. $x^2 + y^2 - 24x + 10y = 0$.

(*c*) center at (0, 0) and radius $= r$. *Ans.* $x^2 + y^2 = r^2$.

(*d*) center at (0, 0) and $r = 5$. *Ans.* $x^2 + y^2 = 25$.

(*e*) center at ($3a$, $4a$) and $r = 5a$.

Ans. $x^2 + y^2 - 2a(3x + 4y) = 0$.

(*f*) center at ($b + c$, $b - c$) and $r = c$.

Ans. $x^2 + y^2 - 2(b + c)x - 2(b - c)y + 2b^2 + c^2 = 0$.

12. Find the equation of a circle whose center is (5, -4) and whose circumference passes through the point (-2, 3).

13. Find the equation of a circle having the line joining (3, -5) and (-2, 2) as a diameter.

14. Find the equation of a circle touching each axis at a distance 6 units from the origin.

15. Find the equation of a circle whose center is the middle point of the line joining (-6, 8) to the origin and whose circumference passes through the point (2, 3).

16. A point moves so that its distances from the two fixed points (2, -3) and (-1, 4) are equal. Find the equation of the locus. *Ans.* $3x - 7y + 2 = 0$.

17. Find the equation of the perpendicular bisector of the line joining

(*a*) (2, 1), (-3, -3). *Ans.* $10x + 8y + 13 = 0$.

(*b*) (3, 1), (2, 4). *Ans.* $x - 3y + 5 = 0$.

(*c*) (-1, -1), (3, 7). *Ans.* $x + 2y - 7 = 0$.

(*d*) (0, 4), (3, 0). *Ans.* $6x - 8y + 7 = 0$.

(*e*) (x_1, y_1), (x_2, y_2).

Ans. $2(x_1 - x_2)x + 2(y_1 - y_2)y + x_2^2 - x_1^2 + y_2^2 - y_1^2 = 0$.

18. Show that in problem 17 the coördinates of the middle point of the line joining the given points satisfy the equation of the perpendicular bisector.

19. Find the equations of the perpendicular bisectors of the sides of the triangle (4, 8), (10, 0), (6, 2). Show that they meet in the point (11, 7).

17. Locus of an equation. The preceding sections have illustrated the fact that a locus problem in Analytic Geometry leads at once to an equation in the variables x and y. This equation having been found or being given, the complete solution of the locus problem requires two things, as already noted in the first section (p. 28) of this chapter, namely,

1. To draw the locus by plotting a sufficient number of points whose coördinates satisfy the given equation, and through which the locus therefore passes.

2. To discuss the nature of the locus, that is, to determine properties of the curve.

These two problems are respectively called:

1. Plotting the locus of an equation (second fundamental problem).

2. Discussing an equation (third fundamental problem).

For the present, then, we concentrate our attention upon some given equation in the variables x and y (one or both) and start out with the definition:

The *locus of an equation* in two variables representing coördinates is the curve or group of curves passing through all points whose coördinates satisfy that equation,* and through such points only.

* An equation in the variables x and y is not necessarily satisfied by the coördinates of any points. For coördinates are *real* numbers, and the form of the equation may be such that it is satisfied by no real values of x and y. For example, the equation

$$x^2 + y^2 + 1 = 0$$

is of this sort, since, when x and y are real numbers, x^2 and y^2 are necessarily positive (or zero), and consequently $x^2 + y^2 + 1$ is always a positive number greater than or equal to 1, and therefore *not* equal to zero. Such an equation therefore has *no locus*. The expression "the locus of the equation is imaginary" is also used.

An equation may be satisfied by the coördinates of a *finite* number of points

From this definition the truth of the following theorem is at once apparent:

Theorem I. *If the form of the given equation be changed in any way (for example, by transposition, by multiplication by a constant, etc.), the locus is entirely unaffected.*

We now take up in order the solution of the second and third fundamental problems.

18. Second fundamental problem.

Rule *to plot the locus of a given equation.*

First step. *Solve the given equation for one of the variables in terms of the other.**

Second step. *By this formula compute the values of the variable for which the equation has been solved by assuming real values for the other variable.*

Third step. *Plot the points corresponding to the values so determined.*†

Fourth step. *If the points are numerous enough to suggest the general shape of the locus, draw a smooth curve through the points.*

Since there is no limit to the number of points which may be computed in this way, it is evident that the locus may be drawn as *accurately as may be desired* by simply plotting a sufficiently large number of points.

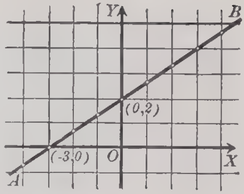
Several examples will now be worked out. The arrangement of the work should be carefully noted.

only. For example, $x^2 + y^2 = 0$ is satisfied by $x = 0$, $y = 0$, but by no other real values. In this case the group of points, one or more, whose coördinates satisfy the equation, is called the locus of the equation.

* The form of the given equation will often be such that solving for one variable is simpler than solving for the other. *Always choose the simpler solution.*

† Remember that *real* values only may be used as coördinates.

EXAMPLES



1. Draw the locus of the equation

$$2x - 3y + 6 = 0.$$

Solution. *First step.* Solving for y,

$$y = \tfrac{2}{3}x + 2.$$

Second step. Assume values for x and compute y, arranging results in the form:

x	y	x	y
0	2	0	2
1	$2\frac{2}{3}$	-1	$1\frac{1}{3}$
2	$3\frac{1}{3}$	-2	$\frac{2}{3}$
3	4	-3	0
4	$4\frac{2}{3}$	-4	$-\frac{2}{3}$
etc.	etc.	etc.	etc.

Thus, if

$$x = 1,\ y = \tfrac{2}{3} \cdot 1 + 2 = 2\tfrac{2}{3},$$
$$x = 2,\ y = \tfrac{2}{3} \cdot 2 + 2 = 3\tfrac{1}{3},$$
etc.

Third step. Plot the points found.

Fourth step. Draw a smooth curve through these points.

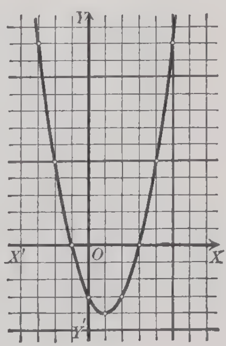
2. Plot the locus of the equation

$$y = x^2 - 2x - 3.$$

Solution. *First step.* The equation as given is solved for y.

Second step. Computing y by assuming values of x, we find the table of values below:

x	y	x	y
0	-3	0	-3
1	-4	-1	0
2	-3	-2	5
3	0	-3	12
4	5	-4	21
5	12	etc.	etc.
6	21		
etc.	etc.		



Third step. Plot the points.

Fourth step. Draw a smooth curve through these points. This gives the curve of the figure.

3. Plot the locus of the equation

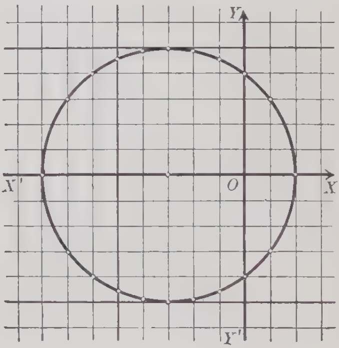
$$x^2 + y^2 + 6x - 16 = 0.$$

Solution. *First step.* Solving for y,

$$y = \pm\sqrt{16 - 6x - x^2}.$$

Second step. Compute y by assuming values of x.

x	y	x	y
0	± 4	0	± 4
1	± 3	-1	± 4.6
2	0	-2	± 4.9
3	imag.	-3	± 5
4	"	-4	± 4.9
5	"	-5	± 4.6
6	"	-6	± 4
7	"	-7	± 3
		-8	0
		-9	imag.



For example, if $x = 1$, $y = \pm\sqrt{16 - 6 - 1} = \pm 3$;

if $x = 3$, $y = \pm\sqrt{16 - 18 - 9} = \pm\sqrt{-11}$,

an imaginary number;

if $x = -1$, $y = \pm\sqrt{16 + 6 - 1} = \pm 4.6$, etc.

Third step. Plot the corresponding points.

Fourth step. Draw a smooth curve through these points.

The student will doubtless remark that the locus of example 1, p. 37, *appears* to be a straight line, and also that the locus of example 3 (above) *appears* to be a circle. This is, in fact, the case. But the *proof* must be reserved for later sections.

PROBLEMS

1. Plot the locus of each of the following equations.

(a) $x+2y=0$.
(b) $x+2y=3$.
(c) $3x-y+5=0$.
(d) $y=4x^2$.
(e) $x^2+4y=0$.
(f) $y=x^2-3$.
(g) $x^2+4y-5=0$.
(h) $y=x^2+x+1$.
(i) $x=y^2+2y-3$.
(j) $4x=y^3$.
(k) $4x=y^3-1$.
(l) $y=x^3-1$.
(m) $y=x^3-x$.
(n) $y=x^3-x^2-5$.
(o) $x^2+y^2=4$.
(p) $x^2+y^2=9$.
(q) $x^2+y^2=25$.
(r) $x^2+y^2+9x=0$.
(s) $x^2+y^2+4y=0$.
(t) $x^2+y^2-6x-16=0$.
(u) $x^2+y^2-6y-16=0$.
(v) $4y=x^4-8$.
(w) $4x=y^4+8$.

2. Show that the following equations have *no locus* (footnote p. 35).

(a) $x^2+y^2+1=0$.
(b) $2x^2+3y^2=-8$.
(c) $x^2+4=0$.
(d) $x^4+y^2+8=0$.
(e) $(x+1)^2+y^2+4=0$.
(f) $x^2+y^2+2x+2y+3=0$.
(g) $4x^2+y^2+8x+5=0$.
(h) $y^4+2x^2+4=0$.
(i) $9x^2+4y^2+18x+8y+15=0$.

Hint. Write each equation in the form of a sum of squares, and reason as in the footnote on p. 35.

The following problems illustrate the

Theorem. *If an equation can be put in the form of a product of variable factors equal to zero, the locus is found by setting each factor equal to zero and plotting each equation separately.*

3. Draw the locus of $\quad 4x^2-9y^2=0.$

Solution. Factoring,

(1) $$(2x-3y)(2x+3y)=0.$$

Then, by the theorem, *the locus consists of the straight lines*

(2) $$2x-3y=0,$$

(3) $$2x+3y=0.$$

Proof. 1. *The coördinates of any point* (x_1, y_1) *which satisfy* (1) *will satisfy either* (2) *or* (3).

For if (x_1, y_1) satisfies (1),

(4) $$(2x_1 - 3y_1)(2x_1 + 3y_1) = 0.$$

This product can vanish only when one of the factors is zero. Hence either

$$2x_1 - 3y_1 = 0,$$

and therefore (x_1, y_1) satisfies (2);

or $$2x_1 + 3y_1 = 0,$$

and therefore (x_1, y_1) satisfies (3).

2. *A point* (x_1, y_1) *on either of the lines defined by* (2) *and* (3) *will also lie on the locus of* (1).

For if (x_1, y_1) is on the line $2x - 3y = 0$,
then (Corollary, p. 30)

(5) $$2x_1 - 3y_1 = 0.$$

Hence the product $(2x_1 - 3y_1)(2x_1 + 3y_1)$ also vanishes, since by (5) the first factor is zero, and therefore (x_1, y_1) satisfies (1).

Therefore every point on the locus of (1) is also on the locus of (2) and (3), and conversely. This proves the theorem for this example. Q.E.D.

4. Show that the locus of each of the following equations is a pair of straight lines, and plot the lines.

(*a*) $x^2 - y^2 = 0$.
(*b*) $9x^2 - y^2 = 0$.
(*c*) $x^2 = 9y^2$.
(*d*) $x^2 - 4x - 5 = 0$.
(*e*) $y^2 - 6y = 7$.
(*f*) $y^2 - 5xy + 6y = 0$.
(*g*) $xy - 2x^2 - 3x = 0$.
(*h*) $xy - 2x = 0$.
(*i*) $xy = 0$.
(*j*) $3x^2 + xy - 2y^2 + 6x - 4y = 0$.
(*k*) $x^2 - y^2 + x + y = 0$.
(*l*) $x^2 - xy + 5x - 5y = 0$.
(*m*) $x^2 - 2xy + y^2 + 6x - 6y = 0$.
(*n*) $x^2 - 4y^2 + 5x + 10y = 0$.
(*o*) $x^2 + 4xy + 4y^2 + 5x + 10y + 6 = 0$.

(p) $x^2+3xy+2y^2+x+y=0.$
(q) $x^2-4xy-5y^2+2x-10y=0.$
(r) $3x^2-2xy-y^2+5x-5y=0.$
(s) $x^2-3xy-4y^2=0.$
(t) $x^2+2xy+y^2+x+y=0.$
(u) $x^2-3xy=0.$
(v) $y^2+4xy=0.$

5. Show that the locus of $Ax^2+Bx+C=0$ is a pair of parallel lines, a single line, or that there is no locus according as $\Delta=B^2-4AC$ is positive, zero, or negative.

6. Show that the locus of $Ax^2+Bxy+Cy^2=0$ is a pair of intersecting lines, a single line, or a point according as $\Delta=B^2-4AC$ is positive, zero, or negative.

19. Third fundamental problem. Discussion of an equation. The method explained of solving the second fundamental problem gives no knowledge of the required curve except that it passes through all the points whose coördinates are determined as satisfying the given equation. Joining these points gives a curve more or less like the exact locus. Serious errors may be made in this way, however, since *the nature of the curve between any two successive points plotted is not determined.* This objection is somewhat obviated by determining *before plotting* certain properties of the locus by a discussion of the given equation now to be explained.

The nature and properties of a locus depend upon the form of its equation, and hence the steps of any discussion must depend upon the particular problem. In every case, however, the following questions should be answered.

1. *Is the curve a closed curve, or does it extend out infinitely far?*

2. *Is the curve symmetrical with respect to either axis or the origin?*

The method of deciding these questions is illustrated in the following examples.

EXAMPLES

1. Plot and discuss the locus of

$$x^2 + 4y^2 = 16. \tag{1}$$

Solution. *First step.* Solving for x,

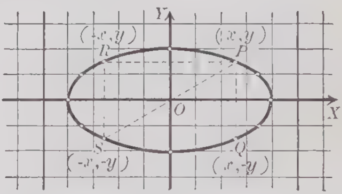
$$x = \pm 2\sqrt{4 - y^2}. \tag{2}$$

Second step. Assume values of y and compute x.

Third step. Plot the points of the table.

Fourth step. Draw a smooth curve through these points.

x	y	x	y
± 4	0	± 4	0
± 3.4	1	± 3.4	-1
± 2.7	$1\frac{1}{2}$	± 2.7	$-1\frac{1}{2}$
0	2	0	-2
imag.	3	imag.	-3



Discussion. 1. Equation (1) shows that neither x nor y can be indefinitely great, since x^2 and $4y^2$ are positive for all real values, and their sum must equal 16. Therefore neither x^2 nor $4y^2$ can exceed 16. Hence *the curve is a closed curve.*

A second way of proving this is the following:

From (2), the ordinate y cannot exceed 2 nor be less than -2, since the expression $4 - y^2$ beneath the radical must not be negative. (2) also shows that x has values only from -4 to 4 inclusive.

2. To determine the symmetry with respect to the axes we proceed as follows:

The equation (1) contains no odd powers of x or y; hence it may be written in any one of the forms

(3) $(x)^2 + 4(-y)^2 = 16$, replacing (x, y) by $(x, -y)$;

(4) $(-x)^2 + 4(y)^2 = 16$, replacing (x, y) by $(-x, y)$;

(5) $(-x)^2 + 4(-y)^2 = 16$, replacing (x, y) by $(-x, -y)$.

The transformation of (1) into (3) corresponds in the figure to replacing each point $P(x, y)$ on the curve by the point $Q(x, -y)$. But the points P and Q are symmetrical with respect to XX', and (1) and (3) have the same locus (Theorem I, p. 36). Hence the locus of (1) is unchanged if each point is changed to a second point symmetrical to the first with respect to XX'. Therefore *the locus is symmetrical with respect to the axis of* x. Similarly, from (4), *the locus is symmetrical with respect to the axis of* y, and from (5), *the locus is symmetrical with respect to the origin.*

The locus is called an **ellipse.**

2. Plot the locus of

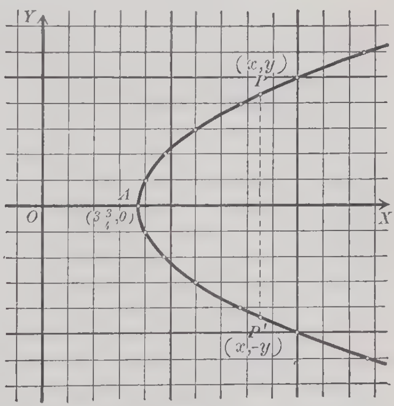
(6) $$y^2 - 4x + 15 = 0.$$

Discuss the equation.

Solution. *First step.* Solve the equation for x, since a square root would have to be extracted if we solved for y. This gives

(7) $$x = \tfrac{1}{4}(y^2 + 15).$$

x	y
$3\frac{3}{4}$	0
4	± 1
$4\frac{3}{4}$	± 2
6	± 3
$7\frac{3}{4}$	± 4
10	± 5
$12\frac{3}{4}$	± 6
etc.	etc.



Second step. Assume values for y and compute x.

Since y^2 only appears in the equation, positive and negative values of y give the same value of x. The calculation gives the table.

For example, if $y = \pm 3,$

then $x = \frac{1}{4}(9 + 15) = 6$, etc.

Third step. Plot the points of the table.

Fourth step. Draw a smooth curve through these points.

Discussion. 1. From (7) it is evident that x increases as y increases. Hence *the curve extends out indefinitely far from both axes.*

2. Since (6) contains no odd powers of y, the equation may be written in the form

$$(-y)^2 - 4(x) + 15 = 0$$

by replacing (x, y) by $(x, -y)$. Hence *the locus is symmetrical with respect to the axis of x.*

The curve is called a **parabola.**

3. Draw the locus of the equation

$$4y = x^3. \tag{8}$$

x	y	x	y
0	0	0	0
1	$\frac{1}{4}$	-1	$-\frac{1}{4}$
$1\frac{1}{2}$	$\frac{27}{32}$	$-1\frac{1}{2}$	$-\frac{27}{32}$
2	2	-2	-2
$2\frac{1}{2}$	$3\frac{29}{32}$	$-2\frac{1}{2}$	$-3\frac{29}{32}$
3	$6\frac{3}{4}$	-3	$-6\frac{3}{4}$
$3\frac{1}{2}$	$10\frac{23}{32}$	$-3\frac{1}{2}$	$-10\frac{23}{32}$

Solution. *First step.* Solving for y,

$$y = \frac{1}{4}x^3.$$

Second step. Assume values for x and compute y. Values of x *must be taken between the integers* in order to give points not too far apart.

For example, if

$$x = 2\frac{1}{2},$$

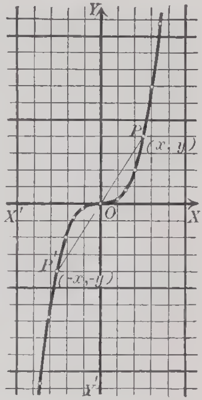
$$y = \frac{1}{4} \cdot \frac{125}{8} = \frac{125}{32} = 3\frac{29}{32}, \text{ etc.}$$

Third step. Plot the points thus found.

Fourth step. The points determine the curve of the figure.

Discussion. 1. From the given equation (8), x and y in-

crease simultaneously, and therefore the curve extends out indefinitely from both axes.



2. In (8) there are no even powers nor constant term, so that by changing signs the equation may be written in the form

$$4(-y)=(-x)^3,$$

replacing (x, y) by $(-x, -y)$.

Hence *the locus is symmetrical with respect to the origin.*

The locus is called a **cubical parabola.**

20. Symmetry. In the above examples we have assumed the definition:

If the points of a curve can be arranged in pairs which are symmetrical with respect to an axis or a point, then the curve itself is said to be *symmetrical* with respect to that axis or point.

The method used for testing an equation for symmetry of the locus was as follows: if (x, y) can be replaced by $(x, -y)$ throughout the equation without affecting the locus, then if (a, b) is on the locus, $(a, -b)$ is also on the locus, and the points of the latter occur in pairs symmetrical with respect to XX', etc. Hence

Theorem II. *If the locus of an equation is unaffected by replacing y by $-y$ throughout its equation, the locus is symmetrical with respect to the axis of x.*

If the locus is unaffected by changing x to $-x$ throughout its equation, the locus is symmetrical with respect to the axis of y.

If the locus is unaffected by changing both x and y to $-x$ and $-y$ throughout its equation, the locus is symmetrical with respect to the origin.

These theorems may be made to assume a somewhat different form if the equation is *algebraic* in x and y. The locus of an

algebraic equation in the variables x and y is called an **algebraic curve.** Then from Theorem II follows

Theorem III. Symmetry of an algebraic curve. *If no odd powers of y occur in an equation, the locus is symmetrical with respect to XX'; if no odd powers of x occur, the locus is symmetrical with respect to YY'. If every term is of even* degree, or every term of odd degree, the locus is symmetrical with respect to the origin.*

21. Further discussion. In this section we treat of three more questions which enter into the discussion of an equation.

Is the origin on the curve?

This question is settled by

Theorem IV. *The locus of an algebraic equation passes through the origin when there is no constant term in the equation.*

Proof. The coördinates (0, 0) satisfy the equation when there is no constant term. Hence the origin lies on the curve (Corollary, p. 30). Q.E.D.

What values of x and y are to be excluded?

Since coördinates are real numbers we have the

Rule *to determine all values of x and y which must be excluded.*

Solve the equation for x in terms of y, and from this result determine all values of y for which the computed value of x will be imaginary. These values of y must be excluded.

Solve the equation for y in terms of x, and from this result determine all values of x for which the computed value of y will be imaginary. These values of x must be excluded.

The **intercepts** of a curve on the axis of x are the abscissas of the points of intersection of the curve and XX'.

The intercepts of a curve on the axis of y are the ordinates of the points of intersection of the curve and YY'.

* The constant term must be regarded as of *even* (zero) degree.

Rule *to find the intercepts.*

Substitute $y=0$, and solve for real values of x. This gives the intercepts on the axis of x.

Substitute $x=0$, and solve for real values of y. This gives the intercepts on the axis of y.

The proof of the rule follows at once from the definitions.

The rule just given explains how to answer the question: What are the intercepts of the locus?

22. Directions for discussing an equation. Given an equation, the following questions should be answered in order before plotting the locus.

1. *Is the origin on the locus?*
2. *Is the locus symmetrical with respect to the axes or the origin?*
3. *What are the intercepts?*
4. *What values of x and y must be excluded?*
5. *Is the curve closed or does it pass off indefinitely far?*

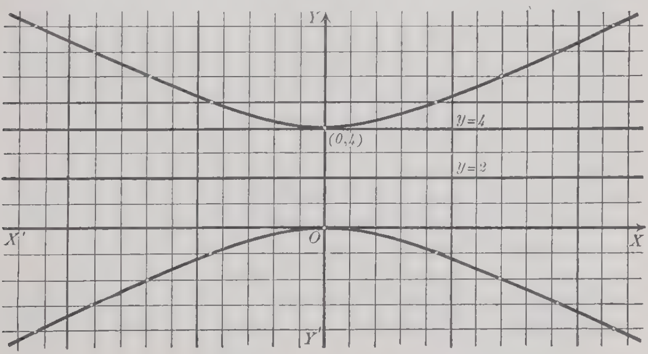
Answering these questions constitutes what is called a **general discussion** of the given equation.

EXAMPLE

Give a general discussion of the equation

(1) $$x^2-4y^2+16y=0.$$

Draw the locus.



1. Since the equation contains no constant term, the origin is on the curve.

2. The equation contains no odd powers of x; hence the locus is symmetrical with respect to YY'.

3. Putting $y=0$, we find $x=0$, the intercept on the axis of x. Putting $x=0$, we find $y=0$ and 4, the intercepts on the axis of y.

4. Solving for x,

$$x = \pm 2\sqrt{y^2 - 4y}. \tag{2}$$

All values of y must be excluded which make the expression beneath the radical sign negative. Now the roots of $y^2 - 4y = 0$ are $y=0$ and $y=4$. For any value of y *between* these roots, $y^2 - 4y$ is negative. For example, $y=2$ gives $4-8=-4$. Hence all values of y between 0 and 4 must be excluded.

Solving for y,

$$y = 2 \pm \tfrac{1}{2}\sqrt{x^2+16}. \tag{3}$$

Hence no value of x is excluded, since x^2+16 is positive for all values of x.

5. From (3), y increases as x increases, and the curve extends out indefinitely far from both axes.

Plotting the locus, using (2), the curve is found to be as in the figure. The curve is a **hyperbola.**

PROBLEMS

1. Give a general discussion of each of the following equations and draw the locus.

(*a*) $x^2 - 4y = 0$.

(*b*) $y^2 - 4x + 3 = 0$.

(*c*) $x^2 + 4y^2 - 16 = 0$.

(*d*) $9x^2 + y^2 - 18 = 0$.

(*e*) $x^2 - 4y^2 - 16 = 0$.

(*f*) $x^2 - 4y^2 + 16 = 0$.

(*g*) $x^2 - y^2 + 4 = 0$.

(*h*) $x^2 - y + x = 0$.

(*i*) $xy - 4 = 0$.

(*j*) $9y + x^3 = 0$.

(*k*) $4x - y^3 = 0$.

(*l*) $6x - y^4 = 0$.

(*m*) $5x - y + y^3 = 0.$

(*n*) $9y^2 - x^3 = 0.$

(*o*) $9y^2 + x^3 = 0.$

(*p*) $x^2 - y^2 + 6x = 0.$

(*q*) $x^2 + 4y^2 + 8y = 0.$

(*r*) $x^2 + 4xy + 5y^2 = 4.$

(*s*) $x^2 + 4xy + y^2 = 3.$

2. Determine the general nature of the locus in each of the following equations by assuming *particular* values for the arbitrary constants, but not *special* values, that is, values which give the equation an added peculiarity.*

(*a*) $y^2 = 2mx.$

(*b*) $x^2 - 2my = m^2.$

(*c*) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$

(*d*) $2xy = a^2.$

(*e*) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$

(*f*) $x^2 - y^2 = a^2.$

(*g*) $x^2 + y^2 = r^2.$

(*h*) $x^2 + y^2 = 2rx.$

(*i*) $x^2 + y^2 = 2ry.$

(*j*) $ay^2 = x^3.$

(*k*) $a^2y = x^3.$

3. Draw the locus of the equation

$$y^2 = (x - a)(x - b)(x - c),$$

(*a*) when $a < b < c.$

(*b*) when $a = b < c.$

(*c*) when $a < b,\ b = c.$

(*d*) when $a = b = c.$

The loci of the equations (*a*) to (*f*) in problem 2 are all of the class known as *conics*, or *conic sections*, — curves following straight lines and circles in the matter of their simplicity.

A **conic section** is the locus of a point whose distances from a fixed point and a fixed line are in a constant ratio.

4. Show that every conic is represented by an equation of the second degree in x and y.

Hint. Take YY' to coincide with the fixed line, and draw XX' through the fixed point. Denote the fixed point by $(p, 0)$ and the constant ratio by e.

Ans. $(1 - e^2)x^2 + y^2 - 2px + p^2 = 0.$

* For example, in (*a*) and (*b*) $m = 0$ is a special value. In fact, in all these examples zero is a special value for any constant.

5. Discuss and plot the locus of the equation of problem 4,

(*a*) when $e = 1$. The conic is now called a *parabola* (see p. 44).

(*b*) when $e < 1$. The conic is now called an *ellipse* (see p. 43).

(*c*) when $e > 1$. The conic is now called a *hyperbola* (see p. 48).

6. A point moves so that the sum of its distances from the two fixed points (3, 0) and (– 3, 0) is constant and equal to 10. What is the locus? *Ans. Ellipse* $16x^2 + 25y^2 = 400$.

7. A point moves so that the difference of its distances from the two fixed points (5, 0) and (– 5, 0) is constant and equal to 8. What is the locus? *Ans. Hyperbola* $9x^2 - 16y^2 = 144$.

23. Points of intersection. If two curves whose equations are given intersect, the coördinates of each point of intersection must satisfy both equations when substituted in them for the variables (Corollary, p. 30). In Algebra it is shown that *all* values satisfying two equations in two unknowns may be found by regarding these equations as simultaneous in the unknowns and solving. Hence the

Rule *to find the points of intersection of two curves whose equations are given.*

Consider the equations as simultaneous in the coördinates, and solve as in Algebra.

Arrange the real solutions in corresponding pairs. These will be the coördinates of all the points of intersection.

Notice that only *real* solutions correspond to common points of the two curves, since coördinates are always real numbers.

EXAMPLES

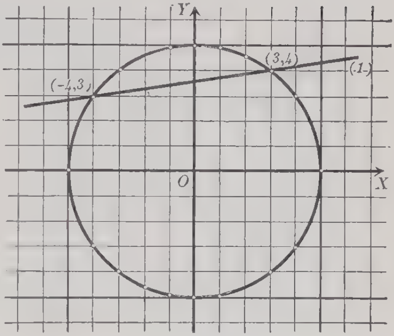
1. Find the points of intersection of

(1) $$x - 7y + 25 = 0,$$

(2) $$x^2 + y^2 = 25.$$

Solution. Solving (1) for x,

(3) $$x = 7y - 25.$$



Substituting in (2),

$$(7y-25)^2+y^2=25.$$

Reducing,

$$y^2-7y+12=0.$$

$$\therefore y=3 \text{ and } 4.$$

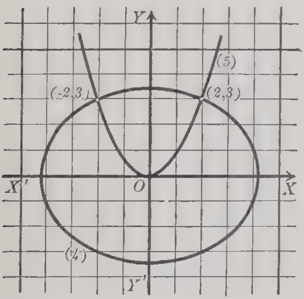
Substituting in (3) [*not* in (2)],

$$x=-4 \text{ and } +3.$$

Arranging, the points of intersection are $(-4, 3)$ and $(3, 4)$. *Ans.*

In the figure the straight line (1) is the locus of equation (1), and the circle the locus of (2).

2. Find the points of intersection of the loci of



(4) $$2x^2+3y^2=35,$$

(5) $$3x^2-4y=0.$$

Solution. Solving (5) for x^2,

(6) $$x^2=\tfrac{4}{3}y.$$

Substituting in (4) and reducing,

$$9y^2+8y-105=0.$$

$$\therefore y=3 \text{ and } -\tfrac{35}{8}.$$

Substituting in (6) and solving,

$$x=\pm 2 \text{ and } \pm\tfrac{1}{6}\sqrt{-210}.$$

Arranging the *real* values, we find the points of intersection are $(+2, 3)$, $(-2, 3)$. *Ans.*

In the figure the ellipse (4) is the locus of (4), and the parabola (5) the locus of (5).

PROBLEMS

Find the points of intersection of the following loci.

1. $\left.\begin{array}{l} 7x-11y+1=0 \\ x+y-2=0 \end{array}\right\}.$

   Ans. $(\frac{7}{6}, \frac{5}{6})$.

2. $\left.\begin{array}{l} x+y=7 \\ x-y=5 \end{array}\right\}.$

   Ans. $(6, 1)$.

3. $\left.\begin{array}{l} y=3x+2 \\ x^2+y^2=4 \end{array}\right\}.$

   Ans. $(0, 2)$, $(-\frac{6}{5}, -\frac{8}{5})$.

4. $\left.\begin{array}{l} y^2=16x \\ y-x=0 \end{array}\right\}.$

   Ans. $(0, 0)$, $(16, 16)$.

5. $\left.\begin{array}{l} x^2+y^2=a^2 \\ 3x+y+a=0 \end{array}\right\}.$

   Ans. $(0, -a), \left(-\frac{3a}{5}, \frac{4a}{5}\right)$.

6. $\left.\begin{array}{l} x^2-y^2=16 \\ x^2=8y \end{array}\right\}.$

   Ans. $(\pm 4\sqrt{2}, 4)$.

7. $\left.\begin{array}{l} x^2+y^2=41 \\ xy=20 \end{array}\right\}.$

   Ans. $(\pm 5, \pm 4)$, $(\pm 4, \pm 5)$.

8. $\left.\begin{array}{l} y^2=2px \\ x^2=2py \end{array}\right\}.$

   Ans. $(0, 0)$, $(2p, 2p)$.

9. $\left.\begin{array}{l} 4x^2+y^2=5 \\ y^2=8x \end{array}\right\}.$

   Ans. $(\frac{1}{2}, 2)$, $(\frac{1}{2}, -2)$.

10. $\left.\begin{array}{l} x^2+y^2=100 \\ y^2=\dfrac{9x}{2} \end{array}\right\}.$

    Ans. $(8, 6)$, $(8, -6)$.

11. $\left.\begin{array}{l} x^2+y^2=5a^2 \\ x^2=4ay \end{array}\right\}.$

    Ans. $(2a, a)$, $(-2a, a)$.

Find the area of the triangles and polygons whose sides are the loci of the following equations.

12. $3x+y+4=0$, $3x-5y+34=0$, $3x-2y+1=0$. *Ans.* 36.

13. $x+2y=5$, $2x+y=7$, $y=x+1$. *Ans.* $\frac{3}{2}$.

14. $x+y=a$, $x-2y=4a$, $y-x+7a=0$. *Ans.* $12a^2$.

15. $x=0$, $y=0$, $x=4$, $y=-6$. *Ans.* 24.

16. $x-y=0$, $x+y=0$, $x-y=a$, $x+y=b$. *Ans.* $\frac{ab}{2}$.

17. $y=3x-9$, $y=3x+5$, $2y=x-6$, $2y=x+14$. *Ans.* 56.

18. Find the distance between the points of intersection of the curves $3x - 2y + 6 = 0$, $x^2 + y^2 = 9$. *Ans.* $\frac{18}{13}\sqrt{13}$.

19. Does the locus of $y^2 = 4x$ intersect the locus of $2x + 3y + 2 = 0$? *Ans.* Yes.

20. For what value of a will the three lines $3x + y - 2 = 0$, $ax + 2y - 3 = 0$, $2x - y - 3 = 0$ meet in a point? *Ans.* $a = 5$.

21. Find the length of the common chord of $x^2 + y^2 = 13$ and $y^2 = 3x + 3$. *Ans.* 6.

22. If the equations of the sides of a triangle are $x + 7y + 11 = 0$, $3x + y - 7 = 0$, $x - 3y + 1 = 0$, find the length of each of the medians. *Ans.* $2\sqrt{5}$, $\frac{5}{2}\sqrt{2}$, $\frac{1}{2}\sqrt{170}$.

CHAPTER IV

STRAIGHT LINE AND CIRCLE

24. The degree of the equation of any straight line. It will now be shown that any straight line is represented by an equation of the first degree in the variable coördinates x and y.

Theorem. *The equation of the straight line passing through a point $B(0, b)$ on the axis of y and having its slope equal to m is*

$$\text{(I)} \qquad \boldsymbol{y = mx + b.}$$

Proof. First step. Assume that $P(x, y)$ is any point on the line.

Second step. The given condition may be written

$$\text{slope of } PB = m.$$

Third step. Since by (II), p. 17,

$$\text{slope of } PB = \frac{y-b}{x-0},$$

[Substituting (x, y) for (x_1, y_1) and $(0, b)$ for (x_2, y_2)]

then
$$\frac{y-b}{x} = m, \text{ or } y = mx + b. \qquad \text{Q.E.D.}$$

In equation (I), m and b may have any values, positive, negative, or zero.

Equation (I) will represent any straight line which intersects the y-axis. But the equation of any line *parallel* to the y-axis has the form $x = a$ *constant*, since the abscissas of all points on such a line are equal. The two forms, $y = mx + b$ and $x =$ constant, will therefore represent all lines. Each of these equations being of the first degree in x and y, we have

Theorem. *The equation of any straight line is of the first degree in the coördinates x and y.*

25. Locus of any equation of the first degree. The question now arises: Given an equation of the first degree in the coördinates x and y, is the locus a straight line?

Consider, for example, the equation

$$(1) \qquad 3x - 2y + 8 = 0.$$

Let us solve this equation for y. This gives

$$(2) \qquad y = \tfrac{3}{2}x + 4.$$

Comparing (2) with the formula (I),

$$y = mx + b,$$

we see that (2) is obtained from (I) if we set $m = \frac{3}{2}$, $b = 4$. Now in (I) m and b may have any values. The locus of (I) is, for all values of m and b, a straight line. Hence (2), or (1), is the equation of a straight line through (0, 4) with the slope equal to $\frac{3}{2}$. This discussion prepares the way for the general theorem.

The equation

$$(3) \qquad Ax + By + C = 0,$$

where A, B, and C are arbitrary constants, is called the **general equation of the first degree** in x and y because every equation of the first degree may be reduced to that form.

Equation (3) represents all straight lines.

For the equation $y = mx + b$ may be written $mx - y + b = 0$, which is of the form (3) if $A = m$, $B = -1$, $C = b$; and the equation $x =$ constant may be written $x -$ constant $= 0$, which is of the form (3) if $A = 1$, $B = 0$, $C = -$ constant.

Theorem. *The locus of the general equation of the first degree*

$$Ax + By + C = 0$$

is a straight line.

Proof. Solving (3) for y, we obtain

$$(4) \qquad y = -\frac{A}{B}x - \frac{C}{B}.$$

Comparison with (I) shows that the locus of (4) is the straight line for which

$$m = -\frac{A}{B},\ b = -\frac{C}{B}.$$

If, however, $B = 0$, the reasoning fails.

But if $B = 0$, (3) becomes

$$Ax + C = 0,$$

or

$$x = -\frac{C}{A}.$$

The locus of this equation is a straight line parallel to the Y-axis. Hence in all cases the locus of (3) is a straight line.

Q.E.D.

Corollary. *The slope of the line*

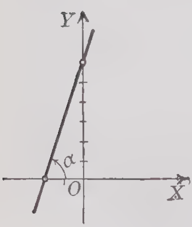
$$Ax + By + C = 0$$

is $m = -\frac{A}{B}$; *that is, the coefficient of* x *with its sign changed divided by the coefficient of* y.

26. Plotting straight lines. If the line does not pass through the origin (constant term not zero, p. 46), find the intercepts (p. 47), mark them off on the axes, and draw the line. If the line passes through the origin, find a second point (p. 37) whose coördinates satisfy the equation.

EXAMPLE

Plot the locus of $3x - y + 6 = 0$. Find the slope.



Solution. Letting $y = 0$ and solving for x, we have

$$x = -2 = \text{intercept on } x\text{-axis}.$$

Letting $x = 0$ and solving for y, we have

$$y = 6 = \text{intercept on } y\text{-axis}.$$

The required line passes through the points $(-2, 0)$ and $(0, 6)$.

To find the slope. Comparison with the general equation (3) shows that $A=3$, $B=-1$, $C=6$. Hence $m=-\frac{A}{B}=3$.

Otherwise thus. Reduce the given equation to the form $y=mx+b$ by solving it for y. This gives $y=3x+6$. Hence $m=3$, $b=6$, as before.

PROBLEMS

1. Find the intercepts and the slope of the following lines and plot the lines.

(*a*) $2x+3y=6$. *Ans.* $3, 2;\ m=-\frac{2}{3}$.

(*b*) $x-2y+5=0$. *Ans.* $-5, 2\frac{1}{2};\ m=\frac{1}{2}$.

(*c*) $3x-y+3=0$. *Ans.* $-1, 3;\ m=3$.

(*d*) $5x+2y-6=0$. *Ans.* $\frac{6}{5}, 3;\ m=-\frac{5}{2}$.

2. Plot the following lines. Find the slope.

(*a*) $2x-3y=0$.

(*b*) $y-4x=0$.

(*c*) $3x+2y=0$.

(*d*) $x-3y=0$.

3. Find the equations, and reduce them to the general form, of the lines for which

(*a*) $m=2$, $b=-3$. *Ans.* $2x-y-3=0$.

(*b*) $m=-\frac{1}{2}$, $b=\frac{3}{2}$. *Ans.* $x+2y-3=0$.

(*c*) $m=\frac{2}{5}$, $b=-\frac{5}{2}$. *Ans.* $4x-10y-25=0$.

(*d*) $\alpha=\frac{\pi}{4}$, $b=-2$. *Ans.* $x-y-2=0$.

(*e*) $\alpha=\frac{3\pi}{4}$, $b=3$. *Ans.* $x+y-3=0$.

Hint. Substitute in $y=mx+b$ and transpose.

4. Select pairs of parallel and perpendicular lines from the following.

(a) $\begin{cases} L_1 : y = 2x - 3. \\ L_2 : y = -3x + 2. \\ L_3 : y = 2x + 7. \\ L_4 : y = \frac{1}{3}x + 4. \end{cases}$ *Ans.* $L_1 \parallel L_3$; $L_2 \perp L_4$.

(b) $\begin{cases} L_1 : x + 3y = 0. \\ L_2 : 8x + y + 1 = 0. \\ L_3 : 9x - 3y + 2 = 0. \end{cases}$ *Ans.* $L_1 \perp L_3$.

(c) $\begin{cases} L_1 : 2x - 5y = 8. \\ L_2 : 5y + 2x = 8. \\ L_3 : 35x - 14y = 8. \end{cases}$ *Ans.* $L_2 \perp L_3$.

5. Show that the quadrilateral whose sides are $2x - 3y + 4 = 0$, $3x - y - 2 = 0$, $4x - 6y - 9 = 0$, and $6x - 2y + 4 = 0$ is a parallelogram.

6. Find the equation of the line whose slope is -2 which passes through the point of intersection of $y = 3x + 4$ and $y = -x + 4$. *Ans.* $2x + y - 4 = 0$.

7. Write an equation which will represent all lines parallel to the line

(a) $y = 2x + 7$. (c) $y - 3x - 4 = 0$.

(b) $y = -x + 9$. (d) $2y - 4x + 3 = 0$.

8. Find the equation of the line parallel to $2x - 3y = 0$ whose intercept on the Y-axis is -2. *Ans.* $2x - 3y - 6 = 0$.

27. Point-slope equation. If it is required that a straight line shall pass through a given point in a given direction, the line is determined.

The following problem is therefore definite:

To find the equation of the straight line passing through a given point $P_1(x_1, y_1)$ and having a given slope m.

Solution. Let $P(x, y)$ be any other point on the line. By the hypothesis,

$$\text{slope } PP_1 = m.$$

(1) $$\therefore \frac{y-y_1}{x-x_1}=m. \qquad \text{(II, p. 17)}$$

Clearing of fractions gives the formula

(II) $$\boldsymbol{y-y_1=m(x-x_1).}$$

28. Two-point equation. A straight line is determined by two of its points. Let us then solve the problem:

To find the equation of the line passing through two given points $P_1(x_1, y_1)$, $P_2(x_2, y_2)$.

Solution. The slope of the given line is

$$\text{slope } P_1P_2=\frac{y_1-y_2}{x_1-x_2}.$$

Let $P(x, y)$ be any other point on the line P_1P_2. Then

$$\text{slope } PP_1=\frac{y-y_1}{x-x_1}.$$

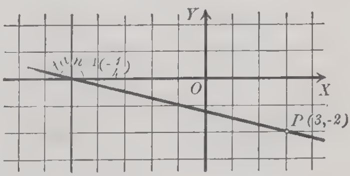
Since P, P_1, and P_2 are on one line, slope $PP_1=$ slope P_1P_2. Hence we have the formula

(III) $$\boldsymbol{\frac{y-y_1}{x-x_1}=\frac{y_1-y_2}{x_1-x_2}.}$$

EXAMPLES

1. Find the equation of the line passing through $P_1(3, -2)$ whose slope is $-\frac{1}{4}$.

Solution. Use the point-slope equation (II), substituting $x_1=3$, $y_1=-2$, $m=-\frac{1}{4}$. This gives



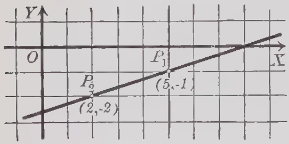
$$y+2=-\tfrac{1}{4}(x-3).$$

Clearing and reducing,

$$x+4y+5=0.$$

2. Find the equation of the line through the two points $P_1(5, -1)$ and $P_2(2, -2)$.

Solution. Use the two-point equation (III), substituting $x_1 = 5$, $y_1 = -1$, $x_2 = 2$, $y_2 = -2$. This gives



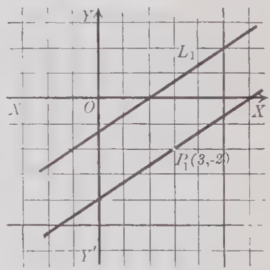
$$\frac{y+1}{x-5} = \frac{-1+2}{5-2} = \frac{1}{3}.$$

Clearing and reducing,

$$x - 3y - 8 = 0.$$

The answer should be *checked*. To do this, we must prove that the coördinates of the given points satisfy the answer. Thus for P_1, substituting $x=5$, $y=-1$, the answer holds. Similarly for P_2. The student should supply checks for examples 1 and 3.

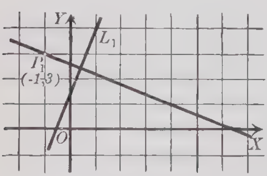
3. Find the equation of the line through the point $P_1(3, -2)$ parallel to the line $L_1: 2x - 3y - 4 = 0$.



Solution. The slope of the given line L_1 equals $\frac{2}{3}$. Hence the slope of the required line also equals $\frac{2}{3}$ (Theorem, p. 18), and it passes through $P_1(3, -2)$. Using the point-slope equation (II), we have

$$y+2=\tfrac{2}{3}(x-3), \text{ or } 2x-3y-12=0.$$

4. Find the equation of the line through the point $P_1(-1, 3)$ perpendicular to the line $L_1: 5x - 2y + 3 = 0$.



Solution. The slope of the given line L_1 equals $\frac{5}{2}$. Hence the slope of the required line equals $-\frac{2}{5}$ (Theorem, p. 18). Since we know a point P_1 $(-1, 3)$ on the line, we use the point-slope equation (II), and obtain

$$y - 3 = -\tfrac{2}{5}(x+1), \text{ or } 2x + 5y - 13 = 0.$$

PROBLEMS

1. Find the equation of the line satisfying the following conditions, and plot the lines. Check the answers.

(*a*) Passing through (0, 0) and (8, 2). *Ans.* $x - 4y = 0$.

(*b*) Passing through $(-1, 1)$ and $(-3, 1)$. *Ans.* $y - 1 = 0$.

(*c*) Passing through $(-3, 1)$ and slope $= 2$.
Ans. $2x - y + 7 = 0$.

(*d*) Having the intercepts* $a = 3$ and $b = -2$.
Ans. $2x - 3y - 6 = 0$.

(*e*) Slope $= -3$, intercept on X-axis $= 4$.
Ans. $3x + y - 12 = 0$.

(*f*) Intercepts $a = -3$ and $b = -4$.
Ans. $4x + 3y + 12 = 0$.

(*g*) Passing through (2, 3) and $(-2, -3)$.
Ans. $3x - 2y = 0$.

(*h*) Passing through (3, 4) and $(-4, -3)$.
Ans. $x - y + 1 = 0$.

(*i*) Passing through (2, 3) and slope $= -2$.
Ans. $2x + y - 7 = 0$.

2. Find the equation of the line passing through the origin parallel to the line $2x - 3y = 4$. *Ans.* $2x - 3y = 0$.

3. Find the equation of the line passing through the origin perpendicular to the line $5x + y - 2 = 0$. *Ans.* $x - 5y = 0$.

4. Find the equation of the line passing through the point (3, 2) parallel to the line $4x - y - 3 = 0$.
Ans. $4x - y - 10 = 0$.

5. Find the equation of the line passing through the point (3, 0) perpendicular to the line $2x + y - 5 = 0$.
Ans. $x - 2y - 3 = 0$.

6. Find the equation of the line whose intercept on the Y-axis is 5 which passes through the point (6, 3).
Ans. $x + 3y - 15 = 0$.

* Intercept on x-axis $= a$, intercept on y-axis $= b$. The given points are (3, 0) and $(0, -2)$.

7. Find the equation of the line whose intercept on the X-axis is 3 which is parallel to the line $x - 4y + 2 = 0$.

Ans. $x - 4y - 3 = 0$.

8. Find the equation of the line passing through the origin and through the intersection of the lines $x - 2y + 3 = 0$ and $x + 2y - 9 = 0$. *Ans.* $x - y = 0$.

9. Find the equations of the sides of the triangle whose vertices are $(-3, 2)$, $(3, -2)$, and $(0, -1)$.

Ans. $2x + 3y = 0$, $x + 3y + 3 = 0$, and $x + y + 1 = 0$.

10. Find the equations of the medians of the triangle in problem 9, and show that they meet in a point.

Ans. $x = 0$, $7x + 9y + 3 = 0$, and $5x + 9y + 3 = 0$.

Hint. To show that three lines meet in a point, find the point of intersection of two of them and prove that it lies on the third.

11. Determine whether or not the following sets of points lie on a straight line.

(*a*) $(0, 0)$, $(1, 1)$, $(7, 7)$. *Ans.* Yes.
(*b*) $(2, 3)$, $(-4, -6)$, $(8, 12)$. *Ans.* Yes.
(*c*) $(3, 4)$, $(1, 2)$, $(5, 1)$. *Ans.* No.
(*d*) $(3, -1)$, $(-6, 2)$, $(-\frac{3}{2}, 1)$. *Ans.* No.
(*e*) $(5, 6)$, $(\frac{5}{6}, 1)$, $(-1, -\frac{6}{5})$. *Ans.* Yes.
(*f*) $(7, 6)$, $(2, 1)$, $(6, -2)$. *Ans.* No.

12. Find the equations of the lines joining the middle points of the sides of the triangle in problem 9, and show that they are parallel to the sides.

Ans. $4x + 6y + 3 = 0$, $x + 3y = 0$, and $x + y = 0$.

13. Find the equation of the line passing through the origin and through the intersection of the lines $x + 2y = 1$ and $2x - 4y - 3 = 0$. *Ans.* $x + 10y = 0$.

14. Show that the diagonals of a square are perpendicular.

Hint. Take two sides for the axes and let the length of a side be a.

15. Show that the line joining the middle points of two sides of a triangle is parallel to the third.

Hint. Choose the axes so that the vertices are $(0, 0)$, $(a, 0)$, and (b, c).

16. Two sides of a parallelogram are $2x+3y-7=0$ and $x-3y+4=0$. Find the other two sides if one vertex is the point $(3, 2)$. *Ans.* $2x+3y-12=0$ and $x-3y+3=0$.

17. Find the equations of the lines drawn through the vertices of the triangle whose vertices are $(-3, 2)$, $(3, -2)$, and $(0, -1)$, which are parallel to the opposite sides.

Ans. The sides of the triangle are

$$2x+3y=0,\ x+3y+3=0,\ x+y+1=0.$$

The required equations are

$$2x+3y+3=0,\ x+3y-3=0,\ x+y-1=0.$$

18. Find the equations of the lines drawn through the vertices of the triangle in problem 17 which are perpendicular to the opposite sides, and show that they meet in a point.

Ans. $3x-2y-2=0$, $3x-y+11=0$, $x-y-5=0$.

19. Find the equations of the perpendicular bisectors of the sides of the triangle in problem 17, and show that they meet in a point. *Ans.* $3x-2y=0$, $3x-y-6=0$, $x-y+2=0$.

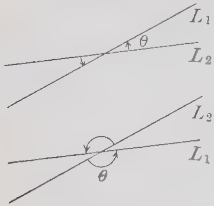
20. The equations of two sides of a parallelogram are $3x-4y+6=0$ and $x+5y-10=0$. Find the equations of the other two sides if one vertex is the point $(4, 9)$.

Ans. $3x-4y+24=0$ and $x+5y-49=0$.

21. The vertices of a triangle are $(2, 1)$, $(-2, 3)$, and $(4, -1)$. Find the equations of (a) the sides of the triangle, (b) the perpendicular bisectors of the sides, and (c) the lines drawn through the vertices perpendicular to the opposite sides. Check the results by showing that the lines in (b) and (c) meet in a point.

29. The angle which a line makes with a second line The angle between two directed lines has been defined (p. 16)

as the angle between their positive directions. When a line is given by means of its equation, no positive direction along the line is fixed. In order to distinguish between the two pairs of equal angles which two intersecting lines make with each other, we define the **angle which a line makes with a second line** to be the positive angle (p. 2) from the *second* line to the *first* line.



Thus the angle which L_1 makes with L_2 is the angle θ. We speak always of the "angle which one line makes with a second line," and the use of the phrase "the angle *between* two lines" should be avoided if those lines are not directed lines.

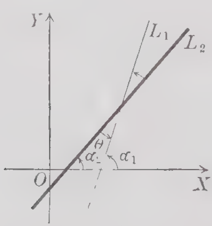
Theorem. *If m_1 and m_2 are the slopes of two lines, then the angle θ which the first line makes with the second is given by*

$$\text{(IV)} \qquad \tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2}.$$

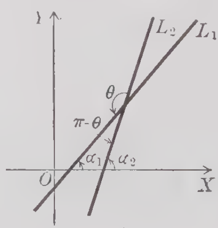
Proof. Let α_1 and α_2 be the inclinations of L_1 and L_2 respectively. Then, since the exterior angle of a triangle equals the sum of the two opposite interior angles, we have

In Fig. 1, $\quad \alpha_1 = \theta + \alpha_2, \quad$ or $\theta = \alpha_1 - \alpha_2$,

In Fig. 2, $\quad \alpha_2 = \pi - \theta + \alpha_1, \quad$ or $\theta = \pi + (\alpha_1 - \alpha_2)$.



(1)



(2)

And since (30, p. -)

$$\tan(\pi + \phi) = \tan\phi,$$

we have, in either case,

$$\tan\theta = \tan(\alpha_1 - \alpha_2)$$

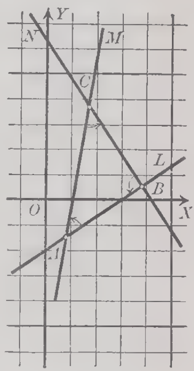
$$= \frac{\tan\alpha_1 - \tan\alpha_2}{1 + \tan\alpha_1 \tan\alpha_2}. \qquad \text{(by 38, p. 3)}$$

But $\tan\alpha_1$ is the slope of L_1, and $\tan\alpha_2$ is the slope of L_2; hence, writing $\tan\alpha_1 = m_1$, $\tan\alpha_2 = m_2$, we have (IV).

In applying (IV), we remember that m_2 = slope of the line *from* which θ is measured in the *positive* direction.

EXAMPLES

1. Find the angles of the triangle formed by the lines whose equations are



$$L: 2x - 3y - 6 = 0,$$
$$M: 6x - y - 6 = 0,$$
$$N: 6x + 4y - 25 = 0.$$

Solution. To see which angles formed by the given lines are the angles of the triangle, we plot the lines, obtaining the triangle ABC.

Let us find the angle A. In the figure, A is measured *from* the line L. Hence in (IV), m_2 = slope of $L = \frac{2}{3}$, m_1 = slope of $M = 6$.

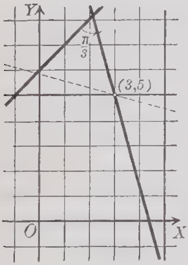
$$\therefore \tan A = \frac{6 - \frac{2}{3}}{1 + 4} = \frac{16}{15}, \text{ and } A = \tan^{-1}\tfrac{16}{15}.$$

Next find the angle at B. In the figure, B is measured *from* N. Hence m_2 = slope of $N = -\frac{3}{2}$, m_1 = slope of $L = \frac{2}{3}$. Hence $m_2 = -\dfrac{1}{m}$, and $B = \dfrac{\pi}{2}$.

Finally, the angle at C is measured *from* the line M. Hence in (IV) m_2 = slope of $M = 6$, m_1 = slope of $N = -\frac{3}{2}$.

$$\therefore \tan C = \frac{-\frac{3}{2} - 6}{1 - 9} = \frac{15}{16}, \text{ and } C = \tan^{-1}\tfrac{15}{16}.$$

We may verify these results. For if $B = \frac{\pi}{2}$, then $A = \frac{\pi}{2} - C$; and hence (31, p. 3, and 26, p. 3) $\tan A = \cot C = \frac{1}{\tan C}$, which is true for the values found.



2. Find the equation of the line through (3, 5) which makes an angle of $\frac{\pi}{3}$ with the line $x - y + 6 = 0$.

Solution. Let m_1 be the slope of the required line. Then its equation is by (II), p. 59,

(1) $$y - 5 = m_1(x - 3).$$

The slope of the given line is $m_2 = 1$, and since the angle which (1) makes with the given line is $\frac{\pi}{3}$, we have,

$$\tan\frac{\pi}{3} = \frac{m_1 - 1}{1 + m_1},$$

or
$$\sqrt{3} = \frac{m_1 - 1}{1 + m_1},$$

whence
$$m_1 = \frac{1 + \sqrt{3}}{1 - \sqrt{3}} = -(2 + \sqrt{3}).$$

Substituting in (1), we obtain

$$y - 5 = -(2 + \sqrt{3})(x - 3),$$

or
$$(2 + \sqrt{3})x + y - (11 + 3\sqrt{3}) = 0.$$

In Plane Geometry there would be two solutions of this problem, — the line just obtained and the dotted line of the figure. Why must the latter be excluded here?

In working out the following problems, the student should first draw the figure and mark by an arc the angle desired, remembering that this angle is measured from the second line to the first in the counter-clockwise direction.

PROBLEMS

1. Find the angle which the line $3x - y + 2 = 0$ makes with $2x + y - 2 = 0$; also the angle which the second line makes with the first, and show that these angles are supplementary.

Ans. $\frac{3\pi}{4}, \frac{\pi}{4}$.

2. Find the angle which the line

(*a*) $2x - 5y + 1 = 0$ makes with the line $x - 2y + 3 = 0$.

(*b*) $x + y + 1 = 0$ makes with the line $x - y + 1 = 0$.

(*c*) $3x - 4y + 2 = 0$ makes with the line $x + 3y - 7 = 0$.

(*d*) $6x - 3y + 3 = 0$ makes with the line $x = 6$.

(*e*) $x - 7y + 1 = 0$ makes with the line $x + 2y - 4 = 0$.

In each case plot the lines and mark the angle found by a small arc.

Ans. (*a*) $\tan^{-1}(-\frac{1}{12})$; (*b*) $\frac{\pi}{2}$; (*c*) $\tan^{-1}(\frac{13}{9})$; (*d*) $\tan^{-1}(-\frac{1}{2})$; (*e*) $\tan^{-1}(\frac{9}{13})$.

3. Find the angles of the riangle whose sides are $x + 3y - 4 = 0$, $3x - 2y + 1 = 0$, and $x - y + 3 = 0$.

Ans. $\tan^{-1}(-\frac{11}{3})$, $\tan^{-1}(\frac{1}{5})$, $\tan^{-1}(2)$.

Hint. Plot the triangle to see which angles formed by the given lines are the angles of the triangle.

4. Find the exterior angles of the triangle formed by the lines $5x - y + 3 = 0$, $y = 2$, $x - 4y + 3 = 0$.

Ans. $\tan^{-1}(5)$, $\tan^{-1}(-\frac{1}{4})$, $\tan^{-1}(-\frac{19}{9})$.

5. Find one exterior angle and the two opposite interior angles of the triangle formed by the lines $2x - 3y - 6 = 0$, $3x + 4y - 12 = 0$, $x - 3y + 6 = 0$. Verify the results by formula 37, p. 3.

6. Find the angles of the triangle formed by $3x + 2y - 4 = 0$, $x - 3y + 6 = 0$, and $4x - 3y - 10 = 0$. Verify the results by the formula

$\tan A + \tan B + \tan C = \tan A \tan B \tan C$, if $A + B + C = 180°$.

7. Find the line passing through the given point and making the given angle with the given line.

(*a*) $(2, 1)$, $\frac{\pi}{4}$, $2x - 3y + 2 = 0$. *Ans.* $5x - y - 9 = 0$.

(*b*) $(1, -3)$, $\frac{3\pi}{4}$, $x + 2y + 4 = 0$. *Ans.* $3x + y = 0$.

(*c*) (x_1, y_1), ϕ, $y = mx + b$. *Ans.* $y - y_1 = \frac{m + \tan\phi}{1 - m\tan\phi}(x - x_1)$.

(*d*) (x_1, y_1), ϕ, $Ax + By + C = 0$.

Ans. $y - y_1 = \frac{B\tan\phi - A}{A\tan\phi + B}(x - x_1)$.

30. Equation of the circle. Every circle is determined when its center and radius are known.

Theorem. *The equation of the circle whose center is a given point (α, β) and whose radius equals r is*

$$(x - \alpha)^2 + (y - \beta)^2 = r^2. \tag{V}$$

Proof. First step. Assume that $P(x, y)$ is any point on the locus.

Second step. If the center (α, β) be denoted by C, the given condition is

$$PC = r.$$

Third step. By (I), p. 13,

$$PC = \sqrt{(x - \alpha)^2 + (y - \beta)^2}.$$

$$\therefore \sqrt{(x - \alpha)^2 + (y - \beta)^2} = r.$$

Squaring, we have (V). Q.E.D.

Corollary. *The equation of the circle whose center is the origin $(0, 0)$ and whose radius is r is*

$$x^2 + y^2 = r^2.$$

If (V) is expanded and transposed, we obtain

$$x^2 + y^2 - 2\alpha x - 2\beta y + \alpha^2 + \beta^2 - r^2 = 0. \tag{1}$$

From the form of this equation we observe:

Any circle is defined by an equation of the *second degree* in the variables x and y, in which *the terms of the second degree consist of the sum of the squares of x and y.*

Equation (1) is of the form

$$x^2 + y^2 + Dx + Ey + F = 0, \tag{2}$$

where

$$D = -2\alpha,\ E = -2\beta, \text{ and } F = \alpha^2 + \beta^2 - r^2. \tag{3}$$

Can we infer, conversely, that the locus of every equation of the form (2) is a circle? By adding $\frac{1}{4}D^2 + \frac{1}{4}E^2$ to both members, (2) becomes

$$(x + \tfrac{1}{2}D)^2 + (y + \tfrac{1}{2}E)^2 = \tfrac{1}{4}(D^2 + E^2 - 4F). \tag{4}$$

In (4) we distinguish three cases:

If $D^2 + E^2 - 4F$ is positive, (4) is in the form (V), and hence the locus of (2) is a circle whose center is $(-\frac{1}{2}D, -\frac{1}{2}E)$ and whose radius is $r = \frac{1}{2}\sqrt{D^2 + E^2 - 4F}$.

If $D^2 + E^2 - 4F = 0$, the only real values satisfying (4) are $x = -\frac{1}{2}D$, $y = -\frac{1}{2}E$ (footnote, p. 35). The locus, therefore, is the single point $(-\frac{1}{2}D, -\frac{1}{2}E)$. In this case the locus of (2) is often called a **point circle,** or a **circle whose radius is zero.**

If $D^2 + E^2 - 4F$ is negative, no real values satisfy (4), and hence (2) has no locus.

The expression $D^2 + E^2 - 4F$ is called the **discriminant** of (2), and is denoted by Θ. The result is given by the

Theorem. *The locus of the equation*

$$\boldsymbol{x^2 + y^2 + Dx + Ey + F = 0,} \tag{VI}$$

whose discriminant is $\Theta = D^2 + E^2 - 4F$, *is determined as follows:*

(*a*) *When Θ is positive the locus is the circle whose center is $(-\frac{1}{2}D, -\frac{1}{2}E)$ and whose radius is $r = \frac{1}{2}\sqrt{D^2 + E^2 - 4F} = \frac{1}{2}\sqrt{\Theta}$.*

(*b*) *When Θ is zero the locus is the point circle $(-\frac{1}{2}D, -\frac{1}{2}E)$.*

(*c*) *When Θ is negative there is no locus.*

Corollary. *When $E=0$ the center of (VI) is on the X-axis, and when $D=0$ the center is on the Y-axis.*

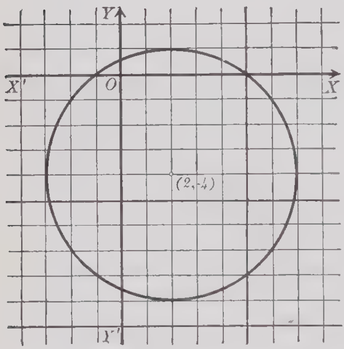
Whenever in what follows it is said that (VI) is the equation of a circle it is assumed that Θ is positive.

EXAMPLE

Find the locus of the equation $x^2+y^2-4x+8y-5=0$.

Solution. The given equation is of the form (VI), where

$$D=-4,\ E=8,\ F=-5,$$



and hence

$$\Theta=16+64+20=100>0.$$

The locus is therefore a circle whose center is the point $(2,-4)$ and whose radius is $\frac{1}{2}\sqrt{100}=5$.

The equation $Ax^2+Bxy+Cy^2+Dx+Ey+F=0$ is called the **general equation of the second degree** in x and y because it contains all possible terms in x and y of the second and lower degrees. This equation can be reduced to the form (VI) when and only when $A=C$ and $B=0$. Hence the locus of an equation of the second degree is a circle only when the coefficients of x^2 and y^2 are equal and the xy-term is lacking.

31. Circles determined by three conditions. The equation of any circle may be written in either one of the forms

$$(x-\alpha)^2+(y-\beta)^2=r^2,$$

or

$$x^2+y^2+Dx+Ey+F=0.$$

Each of these equations contains three arbitrary constants. To determine these constants three equations are necessary, and as any equation between the constants means that the

circle satisfies some geometrical condition, it follows that a circle may be determined to satisfy three conditions.

Rule *to determine the equation of a circle satisfying three conditions.*

First step. *Let the required equation be*

(1) $$(x-\alpha)^2+(y-\beta)^2=r^2,$$

or

(2) $$x^2+y^2+Dx+Ey+F=0,$$

as may be more convenient.

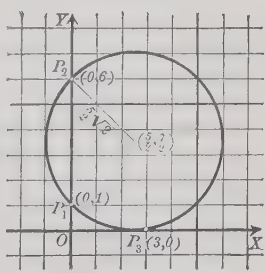
Second step. *Find three equations between the constants* α, β, *and* r [*or* D, E, *and* F] *which express that the circle* (1) [*or* (2)] *satisfies the three given conditions.*

Third step. *Solve the equations found in the second step for* α, β, *and* r [*or* D, E, *and* F].

Fourth step. *Substitute the results of the third step in* (1) [*or* (2)]. *The result is the required equation.*

EXAMPLES

1. Find the equation of the circle passing through the three points $P_1(0, 1)$, $P_2(0, 6)$, and $P_3(3, 0)$.



First solution. *First step.* Let the required equation be

(3) $$x^2+y^2+Dx+Ey+F=0.$$

Second step. Since P_1, P_2, and P_3 lie on (3), their coördinates must satisfy (3). Hence we have

(4) $$1+E+F=0,$$

(5) $$36+6E+F=0,$$

and

(6) $$9+3D+F=0.$$

Third step. Solving (4), (5), and (6), we obtain

$$E = -7,\ F = 6,\ D = -5.$$

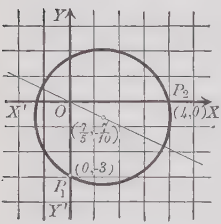
Fourth step. Substituting in (3), the required equation is

$$x^2 + y^2 - 5x - 7y + 6 = 0.$$

The center is $(\frac{5}{2}, \frac{7}{2})$ and the radius is $\frac{5}{2}\sqrt{2} = 3.5$.

Second solution. A second method which follows the geometrical construction for the circumscribed circle is the following. Find the equations of the perpendicular bisectors of P_1P_2 and P_1P_3. The point of intersection is the center. Then find the radius by the length formula.

2. Find the equation of the circle passing through the points $P_1(0, -3)$ and $P_2(4, 0)$ which has its center on the line $x + 2y = 0$.



First solution. *First step.* Let the required equation be

(7) $$x^2 + y^2 + Dx + Ey + F = 0.$$

Second step. Since P_1 and P_2 lie on the locus of (7), we have

(8) $$9 - 3E + F = 0$$

and

(9) $$16 + 4D + F = 0.$$

The center of (7) is $\left(-\frac{D}{2}, -\frac{E}{2}\right)$, and since it lies on the given line,

$$-\frac{D}{2} + 2\left(-\frac{E}{2}\right) = 0,$$

or

(10) $$D + 2E = 0.$$

Third step. Solving (8), (9), and (10), we obtain

$$D = -\tfrac{14}{5},\ E = \tfrac{7}{5},\ \text{and}\ F = -\tfrac{24}{5}.$$

Fourth step. Substituting in (7), we obtain the required equation,

$$x^2 + y^2 - \tfrac{14}{5}x + \tfrac{7}{5}y - \tfrac{24}{5} = 0,$$

or

$$5x^2 + 5y^2 - 14x + 7y - 24 = 0.$$

The center is the point $(\frac{7}{5}, -\frac{7}{10})$, and the radius is $\frac{1}{2}\sqrt{29}$.

Second solution. A second solution is suggested by Geometry, as follows:

Find the equation of the perpendicular bisector of P_1P_2. The point of intersection of this line and the given line is the center of the required circle. The radius is then found by the length formula.

PROBLEMS

1. Find the equation of the circle whose center is

(*a*) $(0, 1)$ and whose radius is 3. *Ans.* $x^2 + y^2 - 2y - 8 = 0$.
(*b*) $(-2, 0)$ and whose radius is 2. *Ans.* $x^2 + y^2 + 4x = 0$.
(*c*) $(-3, 4)$ and whose radius is 5. *Ans.* $x^2 + y^2 + 6x - 8y = 0$.
(*d*) $(\alpha, 0)$ and whose radius is α. *Ans.* $x^2 + y^2 - 2\alpha x = 0$.
(*e*) $(0, \beta)$ and whose radius is β. *Ans.* $x^2 + y^2 - 2\beta y = 0$.
(*f*) $(0, -\beta)$ and whose radius is β. *Ans.* $x^2 + y^2 + 2\beta y = 0$.

2. Find the locus of the following equations.

(*a*) $x^2 + y^2 - 6x - 16 = 0$.
(*b*) $3x^2 + 3y^2 - 10x - 24y = 0$.
(*c*) $x^2 + y^2 = 0$.
(*d*) $x^2 + y^2 - 8x - 6y + 25 = 0$.
(*e*) $x^2 + y^2 - 2x + 2y + 5 = 0$.
(*f*) $x^2 + y^2 - 6x + 4y - 5 = 0$.
(*g*) $(x + 1)^2 + (y - 2)^2 = 0$.
(*h*) $7x^2 + 7y^2 - 4x - y = 3$.
(*i*) $x^2 + y^2 + 2ax + 2by + a^2 + b^2 = 0$.
(*j*) $x^2 + y^2 + 16x + 100 = 0$.

3. Find the equation of the circle which

(*a*) has the center $(2, 3)$ and passes through $(3, -2)$.
Ans. $x^2 + y^2 - 4x - 6y - 13 = 0$.

(*b*) passes through the points $(0, 0)$, $(8, 0)$, $(0, -6)$.
Ans. $x^2 + y^2 - 8x + 6y = 0$.

(*c*) passes through the points $(4, 0)$, $(-2, 5)$, $(0, -3)$.
Ans. $19x^2 + 19y^2 + 2x - 47y - 312 = 0$.

(*d*) passes through the points (3, 5) and (– 3, 7) and has its center on the X-axis. *Ans.* $x^2+y^2+4x-46=0$.

(*e*) passes through the points (4, 2) and (– 6, – 2) and has its center on the Y-axis. *Ans.* $x^2+y^2+5y-30=0$.

(*f*) passes through the points (5, – 3) and (0, 6) and has its center on the line $2x-3y-6=0$.

Ans. $3x^2+3y^2-114x-64y+276=0$.

(*g*) has the center (– 1, – 5) and is tangent to the X-axis.

Ans. $x^2+y^2+2x+10y+1=0$.

(*h*) passes through (1, 0) and (5, 0) and is tangent to the Y-axis. *Ans.* $x^2+y^2-6x\pm 2\sqrt{5}y+5=0$.

(*i*) passes through (0, 1), (5, 1), (2, – 3).

Ans. $2x^2+2y^2-10x+y-3=0$.

(*j*) has the line joining (3, 2) and (– 7, 4) as a diameter.

Ans. $x^2+y^2+4x-6y-13=0$.

(*k*) has the line joining (3, – 4) and (2, – 5) as a diameter.

Ans. $x^2+y^2-5x+9y+26=0$.

(*l*) which circumscribes the triangle formed by $x-6=0$, $x+2y=0$, and $x-2y=8$.

Ans. $2x^2+2y^2-21x+8y+60=0$.

The following problems illustrate cases in which the locus problem is *completely solved* by analytic methods, since the loci may be easily drawn and their nature determined.

LOCUS PROBLEMS

1. Find the equation of the locus of a point whose distances from the axes XX' and YY' are in a constant ratio equal to $\frac{2}{3}$.

Ans. The straight line $2x-3y=0$.

2. Find the equation of the locus of a point the sum of whose distances from the axes of coördinates is always equal to 10.

Ans. The straight line $x+y-10=0$.

3. A point moves so that the difference of the squares of its distances from (3, 0) and (0, – 2) is always equal to 8. Find the equation of the locus, and plot.

Ans. The parallel straight lines $6x+4y+3=0$, $6x+4y-13=0$.

4. A point moves so as to be always equidistant from the axes of coördinates. Find the equation of the locus, and plot.

Ans. The perpendicular straight lines $x+y=0, x-y=0$.

5. A point moves so as to be always equidistant from the straight lines $x-4=0$ and $y+5=0$. Find the equation of the locus, and plot.

Ans. The perpendicular straight lines $x-y-9=0, x+y+1=0$.

6. Find the equation of the locus of a point the sum of the squares of whose distances from $(3, 0)$ and $(-3, 0)$ always equals 68. Plot the locus. *Ans.* The circle $x^2+y^2=25$.

7. Find the equation of the locus of a point which moves so that its distances from $(8, 0)$ and $(2, 0)$ are always in a constant ratio equal to 2. Plot the locus. *Ans.* The circle $x^2+y^2=16$.

8. A point moves so that the ratio of its distances from $(2, 1)$ and $(-4, 2)$ is always equal to $\frac{1}{2}$. Find the equation of the locus, and plot. *Ans.* The circle $3x^2+3y^2-24x-4y=0$.

In the proofs of the following theorems the choice of the axes of coördinates is left to the student, since no mention is made of either coördinates or equations in the problem. In such cases always choose the axes in the most convenient manner possible.

9. A point moves so that the sum of its distances from two perpendicular lines is constant. Show that the locus is a straight line.

Hint. Choosing the axes of coördinates to coincide with the given lines, the equation is $x+y=$ constant.

10. A point moves so that the difference of the squares of its distances from two fixed points is constant. Show that the locus is a straight line.

Hint. Draw XX' through the fixed points, and YY' through their middle point. Then the fixed points may be written $(a, 0)$, $(-a, 0)$, and if the "constant difference" be denoted by k, we find for the locus $4ax=k$ or $4ax=-k$.

11. A point moves so that the sum of the squares of its distances from two fixed points is constant. Prove that the locus is a circle.

Hint. Choose axes as in problem 10.

12. A point moves so that the ratio of its distances from two fixed points is constant. Determine the nature of the locus.

Ans. A circle if the constant ratio is not equal to unity, and a straight line if it is.

13. A point moves so that the square of its distance from a fixed point is proportional to its distance from a fixed line through the fixed point. Show that the locus is a circle.

CHAPTER V

CURVE PLOTTING

32. Asymptotes. The following problems elucidate difficulties arising frequently in drawing the locus of an equation.

EXAMPLES

1. Plot the locus of the equation

(1) $$xy - 2y - 4 = 0.$$

Solution. Solving for y,

(2) $$y = \frac{4}{x-2}.$$

x	y	x	y
0	-2	0	-2
1	-4	-1	$-\frac{4}{3}$
$1\frac{1}{2}$	-8	-2	-1
$1\frac{3}{4}$	-16	-4	$-\frac{2}{3}$
2	∞	-5	$-\frac{4}{7}$
$2\frac{1}{4}$	16	$\vdots$	$\vdots$
$2\frac{1}{2}$	8	-10	$-\frac{1}{3}$
3	4	etc.	etc.
4	2		
5	$\frac{4}{3}$		
6	1		
$\vdots$			
12	0.4		
etc.	etc.		

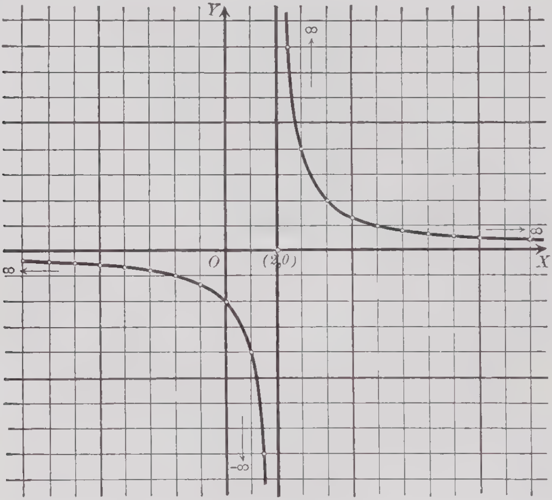
We observe at once, if $x = 2$, $y = \frac{4}{0} = \infty$. This is interpreted thus: The curve *approaches* the line $x = 2$ as it passes off to infinity. The vertical line $x = 2$ is called a *vertical asymptote.*

In plotting, it is necessary to assume values of x differing slightly from 2, both less and greater, as in the table.

From (2) it appears that y diminishes and approaches zero as x increases indefinitely. The curve therefore extends indefinitely far to the right and left, approaching constantly the axis of x. The axis of x is therefore a *horizontal asymptote.* If we solve (1) for x and write the result in the form

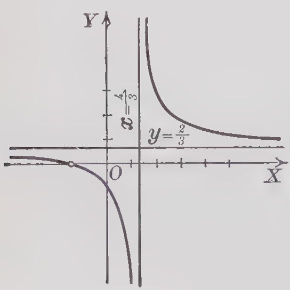
$$x = 2 + \frac{4}{y},$$

it is evident that x approaches 2 as y increases indefinitely. Hence the locus extends both upward and downward indefinitely far, approaching in each case the line $x=2$. This curve is called a *hyperbola.*



In the problem just discussed it was necessary to learn *what value x approached when y became very large,* and also what value y approached when x became very large. These questions, when important, are usually readily answered, as in the following example.

2. Plot the locus of $y=\dfrac{2x+3}{3x-4}$.



When x is very great, we may neglect the 3 in the numerator $(2x+3)$ and the -4 in the denominator $(3x-4)$. That is, when x is very large, $y=\dfrac{2x}{3x}=\dfrac{2}{3}$. Hence $y=\dfrac{2}{3}$ is a horizontal asymptote.

The equation shows directly that $3x-4=0$ or $x=\frac{4}{3}$ is a vertical asymptote. Or we may solve the equation for x which gives

$$x=\frac{4y+3}{3y-2}.$$

Hence, when y is very large, $x=\frac{4y}{3y}=\frac{4}{3}$.

PROBLEMS

Plot each of the following, and determine the horizontal and vertical asymptotes.

1. (a) $xy+y-8=0$.
   (b) $xy+x+3=0$.
   (c) $2xy+2x+3y=0$.
   (d) $x^2+xy+8=0$.
   (e) $2xy+4x-6y+3=0$.
   (f) $y^2+2xy-4=0$.
   (g) $xy+x+2y-3=0$.

2. (a) $x^2y-5=0$.
   (b) $x^2y-y+2x=0$.
   (c) $xy^2-4x+6=0$.
   (d) $x^3y-y+8=0$.
   (e) $y=\frac{5}{x^2-3x}$.
   (f) $y=\frac{4x^2}{x^2-4}$.
   (g) $y=\frac{x-3}{x+1}$.
   (h) $y=\frac{x^2-4}{x^2+x}$.
   (i) $x=\frac{y^2}{y-1}$.
   (j) $x=\frac{y-2}{y-3}$.
   (k) $4x=\frac{y^2}{y^2-9}$.
   (l) $12x=\frac{8y}{3-y^2}$.

33. Natural logarithms. The *common* logarithm of a given number N is the exponent x of the base 10 in the equation

(1) $$10^x=N, \text{ or also } x=\log_{10} N.$$

A second system of logarithms, known as the *natural system*, is of fundamental importance in mathematics. The base of this system is denoted by e, and is called the *natural base*. Numerically to three decimal places,

(2) $$e=2.718.$$

The *natural* logarithm of a given number N is the exponent y in the equation

(3) $$e^y=N, \text{ or also } y=\log_e N.$$

To find the equation connecting the *common* and *natural* logarithms of a given number, we may take the logarithms of both members of (3) to the base 10, which gives

(4) $$\log_{10} e^y = \log_{10} N, \text{ or } y \log_{10} e = \log_{10} N.$$

(5) $$\therefore \log_{10} N = \log_{10} e \cdot \log_e N \text{ [using the value of } y \text{ in (3)].}$$

The equation shows that the common logarithm of any number equals the product of the natural logarithm by the constant $\log_{10} e$. This constant is called the **modulus** ($= M$) of the common system. That is

(6) $$M = \log_{10} e = 0.434. \quad \text{Also } \frac{1}{M} = 2.302.$$

We may summarize in the equations,

(A) **Common log = natural log times 0.434,**
Natural log = common log times 2.302.

Exponential and logarithmic curves. The locus of the equation

(7) $$y = e^x$$

is called an *exponential curve*. To compute values of y, we use logarithms. Taking natural logarithms of both sides,

(8) $$x = \log_e y = 2.302 \log_{10} y.$$

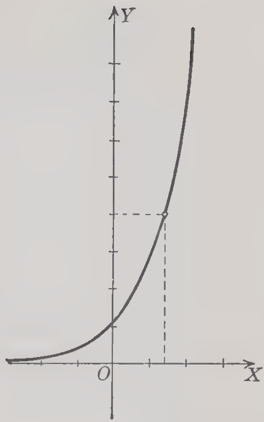
The locus of (7) is therefore the curve whose abscissas are the natural logarithms of the ordinates.

Discussion. Since negative numbers and zero have no logarithms, y is necessarily positive. Moreover, x increases as y increases. The calculation must begin with small values of y, such as $\frac{1}{1000}, \frac{1}{100}, \frac{1}{10}$, these numbers being chosen, since

$$\log_{10} \frac{1}{1000} = \log_{10} \frac{1}{10^3} = \log_{10} 10^{-3} = -3, \text{ etc.} \quad \text{(16 and 19, p. 1)}$$

The computation for determining points on the locus is set down in the table. We use the Table of Art. 2, p. 4. If

x	y	x	y
-9.2	.0001	0	1
-6.9	.001	.693	2
-4.6	.01	1.098	3
-2.3	.1	1.386	4
etc.	etc.	2.302	10
		2.995	20
		4.605	100
		etc.	etc.



the curve is carefully drawn, natural logarithms may be measured off. Thus, by measurement in the figure, if

$$y = 4, \quad x = 1.39.$$

This discussion illustrates the fact that

$$\log_e 0 = -\infty.$$

For clearly as y approaches zero, x becomes *negatively* larger and larger, without limit. Hence the x-axis is a horizontal asymptote.

More generally, the locus of

(9) $$y = e^{kx},$$

where k is a given constant, is an *exponential curve*. The discussion of the difference of this locus from the above figure is left to the reader.

The locus of the equation

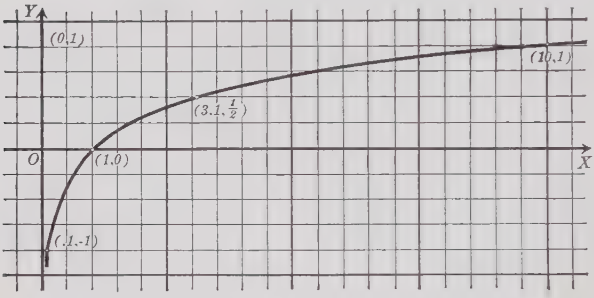
(10) $$y = \log_{10} x,$$

which is called a *logarithmic curve*, differs essentially from the locus of (7) only in its relation to the axes. In fact, both curves are exponential or logarithmic curves, depending upon the point of view.

The locus of (10) is given in the accompanying figure.

Clearly, since $\log_{10} 0 = -\infty$, the y-axis is a vertical asymptote. The scales chosen are

unit length on XX' equals 2 divisions,
unit length on YY' equals 4 divisions.



Compound interest curve. The problem of compound interest introduces exponential curves. For, if $r =$ rate per cent of interest, $n =$ number of years, then the amount ($= A$) of one dollar in n years, if the interest is compounded annually, is given by the formula

$$A = (1 + r)^n.$$

For example, if the rate is 5 per cent, the formula is

(11) $$A = (1.05)^n.$$

If we plot years as abscissas and the amount as ordinates, the corresponding curve will be an exponential curve. For, by Art. 2 and (A),

$$\log_e 1.05 = 2.302 \text{ times } .021$$
$$= .048 \text{ (to three decimal places).}$$

Hence by (3), $e^{.048} = 1.05$, and the equation (11) becomes

(12) $$A = e^{.048\,n},$$

which is in the form (9); that is, $k = .048$.

PROBLEMS

Draw* the loci of each of the following.

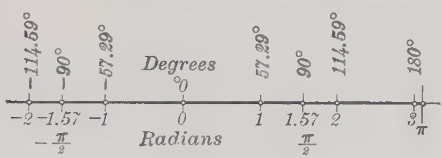
1. $y=e^{-x}$.
2. $y=e^{-\frac{1}{2}x}$.
3. $y=e^{2x}$.
4. $y=e^{-2x}$.
5. $y=2e^{-x}$.
6. $y=2e^{-\frac{1}{4}x}$.
7. $y=xe^{-x}$.
8. $s=t^2e^{-t}$.
9. $v=2e^{-\frac{1}{2}u}$.
10. $y=e^{-x^2}$.
11. $y=2\log_{10}x$.
12. $y=\log_e(1+x)$.
13. $y=2\log_{10}\frac{1}{2}x$.
14. $y=\log_{10}\sqrt{x}$.
15. $y=\log_e(1+e^x)$.
16. $s=\log_{10}(1+2t)$.
17. $v=\log_e(1+t^2)$.
18. $x=\log_{10}(1-y)$.

34. Sine curves. As already explained (p. 2), the two common methods of angular measurement, namely, *circular measure* and *degree measure*, employ as units of measurement the *radian* and the *degree* respectively. The relation between these units is

$$(1)\qquad 1 \text{ radian} = \frac{180}{\pi} \text{ or } 57.29 \text{ degrees},$$

$$1 \text{ degree} = 0.0174 \text{ radians or } \frac{\pi}{180},$$

in which $\pi=3.14$ (or $\frac{22}{7}$ approximately), as usual.



Equations (1) may be written

$$(2)\qquad \pi \text{ radians} = 180 \text{ degrees}.$$

Thus $\frac{\pi}{2}$ radians $=90°$, $\frac{\pi}{4}$ radians $=45°$, etc. The two scales laid off on the same line give the figure.

In advanced mathematics, it is assumed that circular measure is to be used. Thus the numerical values of

* If the *shape* only of the curves 1 – 10 is desired, we may replace e by the approximate value 3 and make the computation without using logarithms.

$$\sin 2x,\ x \tan \frac{\pi x}{4},\ \frac{\cos \frac{\pi x}{6}}{2x}$$

for $x=1$, are as follows:

$$\sin 2x = \sin 2 \text{ radians} = \sin 114°.59 = 0.909,$$

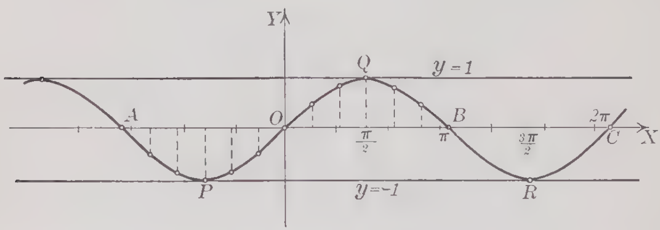
$$x \tan \frac{\pi x}{4} = 1 \cdot \tan\left(\frac{\pi}{4} \text{ radians}\right) = \tan 45° = 1,$$

$$\frac{\cos \frac{\pi x}{6}}{2x} = \frac{\cos\left(\frac{\pi}{6} \text{ radians}\right)}{2} = \frac{\cos 30°}{2} = 0.433.$$

Let us now draw the locus of the equation

$$(3) \qquad y = \sin x,$$

in which, as just remarked, x is the circular measure of an angle.



Solution. Assuming values for x and finding the corresponding number of degrees, we may compute y by the table of Natural Sines, Art. 4.

For example, if

$$x = 1, \text{ since } 1 \text{ radian} = 57°.29,$$

$$y = \sin 57°.29 = .843. \qquad \text{[by (3)]}$$

In making the calculation for plotting, it is convenient to choose angles at intervals of say 30°, and then find x (in radians) and y from the Table of Art. 4.

degrees	x radians	y	degrees	x radians	y
0	0	0	0	0	0
30	.52	.50	$-$ 30	$-$ .52	$-$ .50
60	1.04	.86	$-$ 60	$-$ 1.04	$-$ .86
90	1.56	1.00	$-$ 90	$-$ 1.56	$-$ 1.00
120	2.08	.86	$-$ 120	$-$ 2.08	$-$ .86
150	2.60	.50	$-$ 150	$-$ 2.60	$-$ .50
180	3.14	0	$-$ 180	$-$ 3.14	0

Thus for $30°$, $y = \sin 30° = .50$. For $150°$, $y = \sin 150° = \sin(180° - 30°) = \sin 30° = .50$ (30, p. 3).

The course of the curve beyond B is easily determined from the relation

$$\sin(2\pi + x) = \sin x.$$

Hence
$$y = \sin x = \sin(2\pi + x),$$

that is, the curve is *unchanged if $x + 2\pi$ be substituted for x.* This means, however, that every point is moved a distance 2π to the right. Hence the arc APO may be moved parallel to XX' until A falls on B, that is, into the position BRC, and *it will also be a part of the curve in its new position.* This property is expressed by the statement: The curve $y = \sin x$ is a periodic curve with a **period** equal to 2π. Evidently, the curve crosses OX at intervals equal to a half period. Also, the arc OQB may be displaced parallel to XX' until O falls upon C. In this way it is seen that the entire locus consists of an indefinite number of congruent arcs, alternately above and below XX'.

General discussion. 1. The curve passes through the origin, since (0, 0) satisfies the equation.

2. Since $\sin(-x) = -\sin x$, changing signs in (3),

$$-y = -\sin x,$$

or
$$-y = \sin(-x).$$

Hence the locus is unchanged if (x, y) is replaced by $(-x, -y)$, and the curve is *symmetrical with respect to the origin* (Theorem II, p. 45).

3. In (3), if $x = 0$,

$$y = \sin 0 = 0 = \text{intercept on the axis of } y.$$

Solving (3) for x,

$$x = \sin^{-1} y. \tag{4}$$

In (4), if $y = 0$,

$$x = \sin^{-1} 0$$
$$= n\pi, \ n \text{ being any integer.}$$

Hence the curve cuts the axis of x an indefinite number of times both on the right and left of O, these points being at a distance of π from one another.

4. In (3), x may have any value, since any number is the circular measure of an angle.

In (4), y may have values from -1 to $+1$ inclusive, since the sine of an angle has values only from -1 to $+1$ inclusive.

5. The curve extends out indefinitely along XX' in both directions, but is contained entirely between the lines $y = +1$, $y = -1$.

The locus is called the **wave curve**, from its shape, or the **sine curve**, from its equation (3). The maximum value of y is called the **amplitude**.

Again, let us construct the locus of

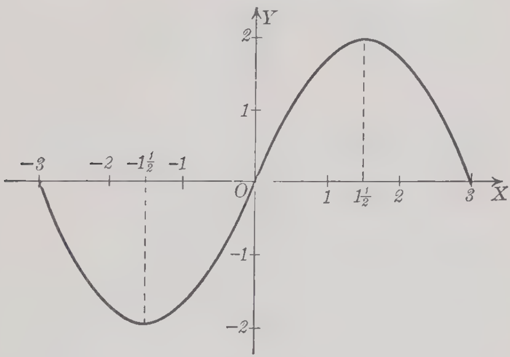
$$y = 2 \sin \frac{\pi x}{3}. \tag{4}$$

Solution. We now choose for x the values 0, $\frac{1}{2}$, 1, $1\frac{1}{2}$, etc., radians, and arrange the work as in the table.

The figure represents a sine curve of period 6 and amplitude 2. For the curve crosses the x-axis at intervals of 3, and the maximum value of y equals 2. To draw any sine curve, after the general shape is known, it is necessary only to find the

x radians	$\frac{1}{3}\pi x$ radians	$\frac{1}{3}\pi x$ degrees	$\sin\frac{\pi x}{3}$	y
0	0	0	0	0
$\frac{1}{2}$	$\frac{1}{6}\pi$	30	.50	1.00
1	$\frac{1}{3}\pi$	60	.86	1.72
$1\frac{1}{2}$	$\frac{1}{2}\pi$	90	1.00	2.00
2	$\frac{2}{3}\pi$	120	.86	1.72
$2\frac{1}{2}$	$\frac{5}{6}\pi$	150	.50	1.00
3	π	180	0	0

amplitude and the period. The maximum values of the ordinate occur at odd multiples of a *quarter* period, and the intersections with OX at multiples of each *half period.*



PROBLEMS

Plot the loci of the equations:*

1. *$y = \cos x$.
2. $y = \sin 2x$.
3. $y = \cos 2x$.
4. $y = \sin \frac{1}{2}x$.
5. $y = \cos \frac{1}{2}x$.
6. $y = \cos \frac{\pi x}{3}$.
7. $y = \sin \frac{\pi x}{4}$.
8. $y = 3\cos \frac{\pi x}{4}$.
9. $y = 2\sin \frac{\pi x}{2}$.

* The cosine curve differs from the sine curve only in the position of the y-axis. The maximum ordinates occur at multiples of half periods and the intersections with OX at odd multiples of quarter periods.

10. $y = 3 \sin \frac{\pi x}{5}$.

11. $y = \tan x$.

12. $y = \tan \frac{\pi x}{4}$.

13. $y = 2 \tan x$.

14. $y = 2 \tan \frac{\pi x}{3}$.

15. $y = 3 \tan \frac{\pi x}{4}$.

16. $y = \cot x$.

17. $y = \cot \frac{\pi x}{4}$.

18. $y = 4 \cot \frac{\pi x}{6}$.

19. $y = \sec x$.

20. $y = \csc x$.

21. $y = \sec \frac{1}{2} x$.

22. $y = \csc \frac{1}{2} x$.

23. $y = \sec \frac{\pi x}{4}$.

24. $y = \csc \frac{\pi x}{4}$.

25. $x = \sin y$. Also written $y = \text{arc} \sin x$ or $\sin^{-1} x$, and read, "the angle whose sine is x."

26. $x = 2 \cos y$, or $y = \text{arc} \cos \frac{1}{2} x$.

27. $x = \tan y$, or $y = \text{arc} \tan x$.

28. $x = 2 \sin \frac{2}{3} \pi y$.

29. $x = \frac{1}{2} \cos \frac{1}{3} \pi y$.

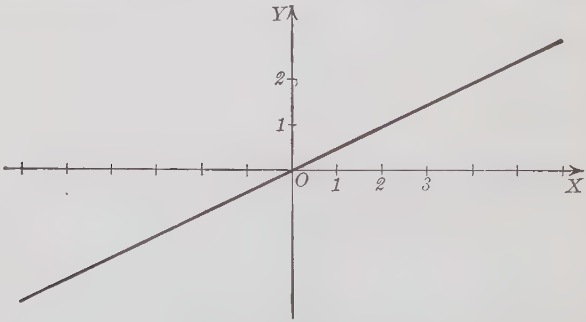
30. $y = \text{arc} \tan \frac{1}{2} x$.

31. $y = 2 \text{ arc} \cos \frac{1}{3} x$.

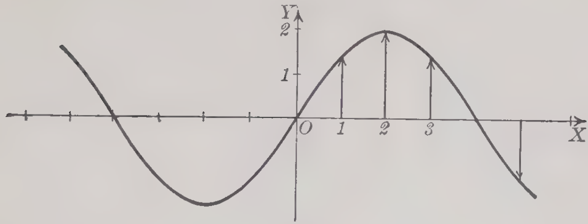
35. Addition of ordinates. When the equation of a curve has the form

$$y = \textit{the algebraic sum of two expressions,}$$

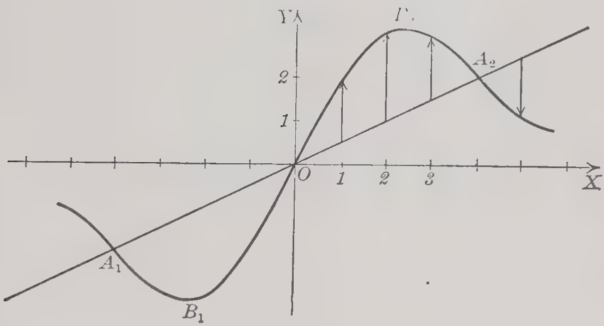
as, for example, $y = \sin x + \cos x$, $y = \frac{1}{2} x + \sin^2 x$, $s = e^t + e^{-t}$, etc., the principle known as addition of ordinates may with advantage be employed. For example, to construct the locus of



(a)



(b)



(c)

(1) $$y=\frac{1}{2}x+2\sin\frac{\pi x}{4},$$

we employ the auxiliary curves

(2) $y_1=\frac{1}{2}x$ (Fig. *a*), $y_2=2\sin\frac{\pi x}{4}$ (Fig. *b*),

using the same axis of ordinates but distinct axes of abscissas. Moreover, the *same scale* must be used in both figures. The ordinates of Fig. *b* are now added (in Fig. *c*) to the corresponding ones in Fig. *a*, attention being given to the algebraic signs. The derived curve $A_1B_1OB_2A_2$ has the equation

(3) $$y=y_1+y_2=\frac{1}{2}x+2\sin\frac{\pi x}{4}$$

as required. The locus winds back and forth across the line $y=\frac{1}{2}x$, crossing this line at $x=0, \pm 4, \pm 8, \pm 12$, etc.

PROBLEMS

Plot the curves:

1. $y = \frac{1}{3}x + \cos x.$

2. $y = \dfrac{x^2}{10} + \sin 2x.$

3. $y = \sin x + \cos x.$

4. $y = \dfrac{1}{4}x - 3\sin\dfrac{\pi x}{3}.$

5. $y = \dfrac{x^2}{16} - 4\cos\dfrac{\pi x}{4}.$

6. $y = \dfrac{e^x + e^{-x}}{2}.$

7. $y = e^x - \sin 2x.$

8. $y = \dfrac{e^t - e^{-t}}{2}.$

9. $y = e^{\frac{x}{4}} - \cos 4x.$

10. $y = \sin x + \sin 2x.$

11. $y = \sin\dfrac{\pi x}{4} + \cos\dfrac{\pi x}{3}.$

12. $y = \sin ax + \cos ax.$

13. $y = 2\sin x + 5\cos x.$

14. $y = 2\sin 2x + 3\cos\frac{1}{2}x$

15. $y = \sin ax + \sin bx.$

16. $y = \frac{1}{4}x\sin x.$

17. $y = x\cos x.$

18. $y = \frac{1}{10}x^2\sin x.$

19. $y = \frac{1}{10}x^2\cos x.$

CHAPTER VI

FUNCTIONS AND GRAPHS

36. Functions. In many practical problems two variables are involved in such a manner that the value of one depends upon the value of the other. For example, given a large number of letters, the postage and the weight are variables, and the amount of the postage depends upon the weight. Again, the premium of a life insurance policy depends upon the age of the applicant. Many other examples will occur to the student.

This relation between two variables is made precise by the definition:

A variable is said to be a function of a second variable when its value depends upon the value of the latter, and is determined when a definite value is assumed for the latter variable.

Thus the *postage* is *determined* when a *definite* weight is assumed; the premium is *determined* when a *definite* age is assumed.

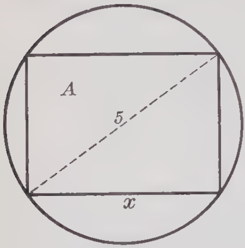
Consider another example:

Draw a circle of diameter 5 in. An indefinite number of rectangles may be inscribed within this circle. But the student will notice that the *entire rectangle* is *determined* as soon as a *side* is drawn. Hence the *area* of the rectangle is a *function* of its side.

Let us now find the equation expressing the relation between a side and the area of the rectangle.

Draw any one of the rectangles and denote the length of its base by x in. Then by drawing a diagonal (which is, of

course, a diameter of the circle), the altitude is found to be equal to $(25-x^2)^{\frac{1}{2}}$. Hence if A denotes the area in *square inches*, we have



(1) $$A = x(25-x^2)^{\frac{1}{2}}.$$

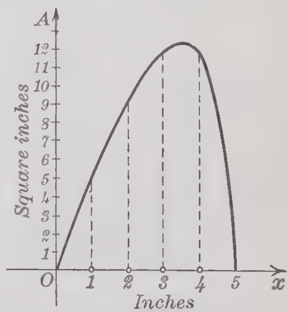
This equation gives the *functional* relation between the function A and the variable x. From it we are enabled to calculate the value of the function A corresponding to any value of the variable x. For example:

if $x=1$ in., $A=(24)^{\frac{1}{2}}=4.9$ sq. in.;
if $x=3$ in., $A=12$ sq. in.;
if $x=4$ in., $A=12$ sq. in.; etc.

To obtain a representation of the equation (1) for *all* values of x, we draw a *graph* of the equation. This we do by drawing rectangular axes and plotting

the values of the variable (x) *as abscissas,*
the values of the function (A) *as ordinates.*

Any functional relation may be *graphed* in this way. We must, however, first *discuss* the equation (1).



The values of x and A are *positive* from the nature of the problem.

The values of x range from zero to 5, inclusive.

The student should now choose a suitable scale on *each* axis and draw the graph. In this case, unit length on the axis of abscissas represents 1 in., and unit length on the axis of ordinates represents 1 sq. in. These two unit lengths need not be the same.

What do we learn from the graph?

1. If carefully drawn, we may *measure from the graph* the *area* of the inscribed rectangle corresponding to any side we choose to assume.

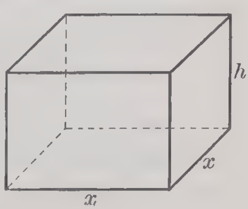
2. There is *one horizontal* tangent. The ordinate at its point of contact is greater than any other ordinate. Hence this discovery: *One of the inscribed rectangles is greater in area than any of the others,* — that is, there is a **maximum** *rectangle.* In other words, the function defined by equation (1) has a **maximum value.**

We cannot, of course, find this value exactly by measurement. For this purpose Calculus is necessary.

The fact that a maximum rectangle *exists* can be seen in advance by reasoning thus: Let the base x increase from zero to 5 in. The area A will then begin with the value zero and return to zero. Since A is always positive, the graph must have a "highest point." Hence there is a maximum value of A, and therefore a maximum rectangle.

Take one more example: A wooden box, open at the top, is to be built to contain 108 cu. ft. The base must be square. This is the only condition. It is evident that under this condition any number of such boxes may be built, and that the number of square feet of lumber used will vary accordingly. If, however, we *choose* any length for a side of the square base, only one box with this dimension can be built, and the material used is determined. *Hence the material used is a function of a side of the square base.*

Let us now find the functional relation between the number of square feet of lumber necessary and the length of one side of the square base measured in feet.



Consider any one box.

Let $M =$ amount of lumber in square feet;

let $x =$ length of side of square base in feet;

let $h =$ height of the box in feet.

Then area base $= x^2$ sq. ft.;

then area sides $= 4\,hx$ sq. ft.

Hence $M = x^2 + 4\,hx.$

But a relation exists between h and x, for the value of M must depend upon the value of x alone. In fact, the volume equals 108 cu. ft.

Hence $hx^2 = 108,$ and $h = \frac{108}{x^2}.$

Therefore

$$M = x^2 + \frac{432}{x}. \tag{2}$$

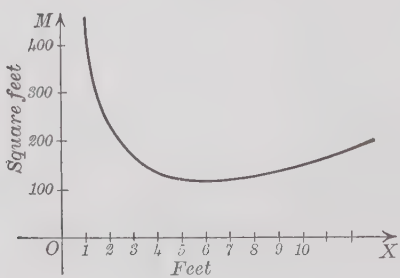
This equation enables us to calculate the number of square feet of lumber in any box with a given square base which has a capacity of 108 cu. ft. The calculation is given in the table:

x	0	1	2	3	4	5	6	7	8	$\cdots$	20	etc.	feet
M	∞	433	220	153	124	111	108	111	118	$\cdots$	421	etc.	sq. ft.

Thus, if $x = 1$ ft., $M = 433$ sq. ft.;

if $x = 4$ ft., $M = 124$ sq. ft.;

if $x = 8$ ft., $M = 118$ sq. ft.; etc.



The student should now graph equation (2), choosing units thus:

unit length on the axis of abscissas represents 1 ft.;
unit length on the axis of ordinates represents 1 sq. ft.

We must, however, choose a very small unit ordinate, since the values of M are large.

A preliminary discussion of (2) shows that x may have any value (positive).

What do we learn from the graph?

(1) If carefully drawn, we may *measure from the graph* the number of square feet of lumber in any box which contains 108 cu. ft. and has a square base.

(2) There is *one horizontal* tangent. The ordinate at its point of contact is less than any other ordinate. Hence this discovery: *One of the boxes takes less lumber than any other;* that is, M has a **minimum** value. This point on the graph can be determined exactly by the Calculus, but careful measurement will in this case give the correct values, viz. $x = 6$, $M = 108$. That is, the construction will take the least lumber (108 sq. ft.) if the base is 6 ft. square.

The fact that a least value of M must exist is seen thus. Let the base increase from a very small square to a very large one. In the former case the height must be very great, and hence the amount of lumber will be large. In the latter case, while the height is small, the base will take a great deal of lumber. Hence M varies from a large value to another large value, and the graph must have a "lowest point."

In the following problems the student will work out the functional relation, draw the graph, and state any conclusions to be drawn from the figure. Care should be exercised in the selection of suitable scales on the axes, especially in the scale adopted for plotting values of the function (compare p. 94). The graph should be neither very flat nor very steep. To avoid the latter we may select a large unit of length for the variable. The plot should be *accurate* so that the maximum and minimum values of the function may be measured.

PROBLEMS

1. Rectangles are inscribed in a circle of radius r. Plot the perimeter P of the rectangles as a function of the breadth x.

Ans. $P = 2x + 2(4r^2 - x^2)^{\frac{1}{2}}$.

2. Right triangles are constructed on a line of a given length h as hypotenuse. Plot (a) the area A and (b) the perimeter P as a function of the length x of one leg.

Ans. (a) $A = \frac{1}{2}x(h^2 - x^2)^{\frac{1}{2}}$. (b) $P = x + h + (h^2 - x^2)^{\frac{1}{2}}$.

3. Right cylinders * are inscribed in a sphere of radius r. Plot as functions of the altitude x of the cylinder, (a) volume V of the cylinder, (b) curved surface S.

Ans. (a) $V = \frac{\pi}{4}(4r^2x - x^3)$. (b) $S = \pi x(4r^2 - x^2)^{\frac{1}{2}}$.

4. Right cones * are inscribed in a sphere of radius r. Plot as functions of the altitude x of the cone, (a) volume V of the cone, (b) curved surface S.

Ans. (a) $V = \frac{\pi}{3}(2rx^2 - x^3)$. (b) $S = \pi(4r^2x^2 - 2rx^3)^{\frac{1}{2}}$.

5. Right cylinders are inscribed in a given right cone. If the height of the cone is h, and the radius of the base r, plot (a) the volume V of the cylinder, (b) the curved surface S, (c) the entire surface T, as functions of the altitude x of the cylinder.

Ans. (a) $V = \frac{\pi r^2 x}{h^2}(h - x)^2$; (b) $S = \frac{2\pi r x}{h}(h - x)$;

(c) $T = \frac{2\pi r}{h^2}(h - x)(rh + (h - r)x)$.

6. Right cones are circumscribed about a sphere of radius r. Plot as a function of the altitude x of the cone, the volume V of the cone.

Ans. $V = \frac{1}{3}\pi\frac{r^2x^2}{x - 2r}$.

* Use formulas 5–9, p. 1.

7. Right cones are constructed with a given slant height L. Plot as functions of the altitude x of the cone, (a) the volume V of the cone, (b) the curved surface S, (c) the entire surface T.

Ans. (a) $V = \frac{1}{3}\pi(L^2x - x^3)$;
(b) $S = \pi L(L^2 - x^2)^{\frac{1}{2}}$.

8. A conical tent is to be constructed of given volume V. Plot the amount A of canvas required as a function of the radius x of the base. *Ans.* $A = \dfrac{(\pi^2x^6 + 9\,V^2)^{\frac{1}{2}}}{x}$.

9. A cylindrical tin can is to be constructed of given volume V. Plot the amount A of tin required as a function of the radius x of the can. *Ans.* $A = 2\,\pi x^2 + \dfrac{2\,V}{x}$.

10. An open box is to be made from a sheet of pasteboard 12 in. square, by cutting equal squares from the four corners and bending up the sides. Plot the volume V as a function of the side x of the square cut out. *Ans.* $V = x(12 - 2\,x)^2$.

11. The strength of a rectangular beam is proportional to the product of the cross section by the square of the depth. Plot the strength S as a function of the depth x for beams which are cut from a log 12 in. in diameter.

Ans. $S = kx^3(144 - x^2)^{\frac{1}{2}}$.

12. A rectangular stockade is to be built to contain a certain area A. A stone wall already constructed is available for one of the sides. Plot the length L of the wall to be built as a function of the length x of the side of the rectangle parallel to the wall. *Ans.* $L = \dfrac{2\,A}{x} + x$.

13. A tower is 100 ft. high. Plot the angle y subtended by the tower at a point on the ground as a function of the distance x from the foot of the tower. *Ans.* $y = \tan^{-1}\dfrac{100}{x}$.

14. A tower 50 ft. high is surmounted by a statue 10 ft. high. If an observer's eyes are in a horizontal plane with the base, plot the angle y subtended by the statue as a function of the observer's distance x from the tower.

$$\textit{Ans.}\quad y = \tan^{-1}\frac{60}{x} - \tan^{-1}\frac{50}{x}.$$

15. A line is drawn through a fixed point (a, b). Plot as a function of the intercept on XX' $(= x)$ of the line, the area A of the triangle formed with the coördinate axes.

$$\textit{Ans.}\quad A = \frac{bx^2}{2(x-a)}.$$

16. A ship is 41 mi. due north of a second ship. The first sails south at the rate of 8 mi. an hour, the second east at the rate of 10 mi. an hour. Plot their distance d apart as a function of the time t which has elapsed since they were in the position given. *Ans.* $d = (164\,t^2 - 656\,t + 1681)^{\frac{1}{2}}$.

17. Plot the distance e from the point $(4, 0)$ to the points (x, y) on the parabola $y^2 = 4\,x$. *Ans.* $e = (x^2 - 4\,x + 16)^{\frac{1}{2}}$.

18. A gutter is to be constructed whose cross section is a broken line made up of three pieces, each 4 inches long, the middle piece being horizontal, and the two sides being equally inclined. Plot the area A of a cross section of the gutter as a function of the width x of the gutter across the top.

$$\textit{Ans.}\quad A = \tfrac{1}{4}(x+4)(48 + 8\,x - x^2)^{\frac{1}{2}}.$$

19. A Norman window consists of a rectangle surmounted by a semicircle. Given the perimeter P, plot the area A as a function of the width x. *Ans.* $A = \dfrac{1}{2}\,xP - \dfrac{1}{2}\,x^2 - \dfrac{\pi}{8}\,x^2$.

20. A person in a boat 9 mi. from the nearest point of the beach wishes to reach a place 15 mi. from that point along the shore. He can row at the rate of 4 mi. an hour and walk at the rate of 5 mi. an hour. The time it takes him to reach his destination depends on the place at which he lands.

Plot the time as a function of the distance x of his landing place from the nearest point on the beach.

$$\textit{Ans.}\ \text{Time} = \frac{\sqrt{81+x^2}}{4} + \frac{15-x}{5}.$$

21. The illumination of a plane surface by a luminous point varies directly as the cosine of the angle of incidence, and inversely as the square of the distance from the surface. Plot the illumination I of a point on the floor 10 ft. from the wall, as a function of the height x of a gas burner on the wall.

$$\textit{Ans.}\ I = \frac{kx}{(100+x^2)^{\frac{3}{2}}}.$$

22. A Gothic window has the shape of an equilateral triangle mounted on a rectangle. The base of the triangle is a chord of the window. The total length of the frame of the window is constant. Express, plot, and discuss the area of the window as a function of the width.

23. A printed page is to contain 24 sq. in. of printed matter. The top and bottom margins are each $1\frac{1}{2}$ in., the side margins 1 in. each. Express, plot, and discuss the area of the page as a function of the width.

24. A manufacturer has 96 sq. ft. of lumber with which to make a box with a square base and a top. Express, plot, and discuss the contents of the box as a function of the side of the base.

25. Isosceles triangles of the same perimeter, 12 in., are cut out of rubber. Express, plot, and discuss the area as a function of the base.

26. Small cylindrical boxes are made each with a cover whose breadth and height are equal. The cover slips on tight. Each box is to hold π cu. in. Express, plot, and discuss the amount of material used as a function of the length of the box.

27. A circular filter paper has a diameter of 11 in. It is folded into a conical shape. Express the volume of the cone

as a function of the angle of the sector folded over. Plot and discuss this function.

28. Two sources of heat are at the points A and B. Remembering that the intensity of heat at a point varies inversely as the square of the distance from the source, express the intensity of heat at any point between A and B as a function of its distance from A. Plot and discuss this function.

29. A submarine telegraph cable consists of a central circular part, called the core, surrounded by a ring. If x denotes the ratio of the radius of the core to the thickness of the ring, it is known that the speed of signaling varies as $x^2 \log \frac{1}{x}$. Plot and discuss this function.

30. A wall 10 ft. high surrounds a square house which is 15 ft. from the wall. Express the length of a ladder placed without the wall, resting upon it and just reaching the house, as a function of the distance of the foot of the ladder from the wall. Plot and discuss this function.

37. Notation of functions. The symbol $f(x)$ is used to denote a function of x, and is read *f of x*. In order to distinguish between different functions, the prefixed letter is changed, as $F(x)$, $\phi(x)$, $f'(x)$, etc.

During any investigation the same functional symbol always indicates the same law of dependence of the function upon the variable. In the simpler cases, this law takes the form of a series of analytical operations upon that variable. Hence, in such a case, the same functional symbol will indicate the same operations or series of operations, even though applied to different quantities. Thus, if

$$f(x) = x^2 - 9x + 14,$$

then $$f(y) = y^2 - 9y + 14.$$

Also $$f(a) = a^2 - 9a + 14,$$

$$f(b+1) = (b+1)^2 - 9(b+1) + 14 = b^2 - 7b + 6,$$

$$f(0)=0^2-9\cdot 0+14=14,$$
$$f(-1)=(-1)^2-9(-1)+14=24,$$
$$f(3)=3^2-9\cdot 3+14=-4,$$
$$f(7)=7^2-9\cdot 7+14=0, \text{ etc.}$$

PROBLEMS

1. Given $\phi(x)=\log_{10} x$. Find $\phi(2)$, $\phi(1)$, $\phi(5)$, $\phi(a-1)$, $\phi(b^2)$, $\phi(x+1)$, $\phi(\sqrt{x})$.

2. Given $\phi(x)=e^{2x}$. Find $\phi(0)$, $\phi(1)$, $\phi(-1)$, $\phi(2y)$, $\phi(-x)$.

3. Given $f(x)=\sin 2x$. Find $f\left(\frac{\pi}{2}\right)$, $f\left(\frac{\pi}{4}\right)$, $f(-\pi)$, $f(-x)$, $f(\pi-x)$, $f(\frac{1}{2}\pi-A)$, $f(\frac{3}{2}\pi+B)$.

4. Given $\theta(x)=\cos x$. Prove

$$\theta(x)+\theta(y)=2\,\theta\left(\frac{x+y}{2}\right)\theta\left(\frac{x-y}{2}\right).$$

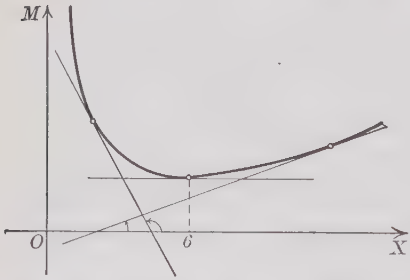
CHAPTER VII

DIFFERENTIATION

38. Tangent at a point on the graph. A glance at the figure on p. 94, that is, the graph of the equation

$$(1) \qquad M = x^2 + \frac{432}{x},$$

in which M represents the number of square feet of lumber required to construct an open box with a square base to contain 108 cu. ft., makes clear the following facts.



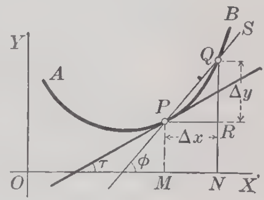
When the base is less than 6 ft. square, the material *decreases* as the size of the base *increases*. (For the graph is *falling* to the *left* of $x = 6$.) When the base is more than 6 ft. square, the material *increases* as the base increases. (For the graph is *rising* to the *right* of $x = 6$.)

If we draw the tangent to the graph at any point to the left of $x = 6$, the slope is clearly negative, while for any point to the right of $x = 6$, the slope is positive. At the lowest point $x = 6$, the slope is zero. Clearly, then, the facts described are characterized by the slope of the tangent to the graph. We may state in general:

If the slope of the graph* is *positive*, then the function *increases* as the variable increases. If the slope of the graph is

* The slope of the graph is the same as the slope of the tangent.

negative, then the function *decreases* as the variable increases. The slope of the graph is *zero* at a maximum or minimum.



39. Differentiation. The discussion just given shows plainly that a method for obtaining the slope of the graph will be useful. We turn our attention to this question.

Let the figure be the graph of the equation

(1) $$y = f(x).$$

It is required to find the slope of the tangent at the point $P(x, y)$.

First step. Take a second point Q on the curve near P, where coördinates * are $(x + \Delta x,\ y + \Delta y)$. These coördinates satisfy equation (1).

Second step. Draw PR parallel to OX. Then $RQ = \Delta y =$ increment of the function. Also, $PR = \Delta x =$ increment of variable.

Third step. Draw the secant through P and Q, and call its inclination ϕ. Then, clearly,

$$\tan\phi = \tan \text{ angle } RPQ = \frac{RQ}{PR},$$

(2) $$\therefore \tan\phi = \frac{\Delta y}{\Delta x} = \frac{\text{increment of function}}{\text{increment of variable}}.$$

Fourth step. Now let the point Q move along the curve toward P as a limiting position. Then

(*a*) the secant turns about P and approaches the tangent at P as a limiting position;

(*b*) the inclination ϕ approaches τ as a limit;

* Δx (read "delta x") is clearly the *change* or increment in the value of x. Similarly, Δy is the increment of y.

(c) the slope of the secant ($=\tan\phi$) approaches the slope of the tangent ($=\tan\tau$) as a limit.

But Q will approach P along the curve if we simply require that Δx *shall vary and approach the limiting value zero.* We thus obtain from (2),

$$\tan\tau = \lim_{\Delta x = 0} \frac{\Delta y}{\Delta x} = \textit{slope of tangent.} \tag{3}$$

The *analytical steps* which parallel those just given lead to the important

General Rule for Differentiation

First step. *In the function replace x by $x+\Delta x$, giving a new value of the function, $y+\Delta y$.*

Second step. *Subtract the given value of the function from the new value in order to find Δy (the increment of the function).*

Third step. *Divide the remainder Δy (the increment of the function) by Δx (the increment of the independent variable).*

Fourth step. *Find the limit of this quotient, when Δx (the increment of the independent variable) varies and approaches the limit zero.*

Let us apply this rule to the equation, p. 102,

$$M = x^2 + \frac{432}{x},$$

in which M takes the place of y.

First step.

$$M + \Delta M = (x+\Delta x)^2 + \frac{432}{x+\Delta x}$$

$$= x^2 + 2x\cdot\Delta x + (\Delta x)^2 + \frac{432}{x+\Delta x}.$$

Second step.

$$M \qquad = x^2 \qquad\qquad\qquad + \frac{432}{x}.$$

$$\Delta M = 2x\cdot\Delta x + (\Delta x)^2 + \frac{432}{x+\Delta x} - \frac{432}{x}$$

$$= (2x+\Delta x)\cdot\Delta x - \frac{432\,\Delta x}{x^2 + x\cdot\Delta x}.$$

Third step. $$\frac{\Delta M}{\Delta x} = 2x + \Delta x - \frac{432}{x^2 + x \cdot \Delta x}.$$

Fourth step. $$\underset{\Delta x = 0}{\text{limit}} \frac{\Delta M}{\Delta x} = 2x - \frac{432}{x^2} = \frac{2x^3 - 432}{x^2}. \quad \textit{Ans.}$$

Hence, by equation (3),

$$(4) \qquad m = \text{slope of tangent at } (x, y) = \frac{2x^3 - 432}{x^2}.$$

Clearly, from this equation, which we may write

$$m = \frac{2(x^3 - 216)}{x^2} = \frac{2(x^3 - 6^3)}{x^2},$$

we see, if $x < 6$, $m < 0$; if $x > 6$, $m > 0$; if $x = 6$, $m = 0$, results agreeing with the figure.

40. Derivative of a function. The general rule for differentiation gives for any function y of x the value of

$$(1) \qquad \underset{\Delta x = 0}{\text{limit}} \frac{\Delta y}{\Delta x}.$$

This result is called the *derivative* of the function with respect to the variable. In words *the derivative of a function is* **the limit** *of the quotient of the increment of the function by the increment of the variable, when the latter increment varies and approaches the limit zero.*

It is customary to use as an abbreviation of the expression (1) the symbol $\frac{dy}{dx}$ (read "*derivative of y with respect to x*"); that is, we place for convenience

$$(2) \qquad \frac{dy}{dx} = \underset{\Delta x = 0}{\text{limit}} \frac{\Delta y}{\Delta x}.$$

The symbol $\frac{dy}{dx}$ is for the present to be regarded as a *whole*, not as a fraction. In a later section we define dy and dx separately and also $\frac{dy}{dx}$ as a fraction.

Similarly, if s is a function of t, then

$$\frac{ds}{dt} = \underset{\Delta t = 0}{\text{limit}} \frac{\Delta s}{\Delta t} = \text{derivative of } s \text{ with respect to } t.$$

It is useful to write down *symbolically* the results of applying the *General Rule* to the general equation

$$(3) \qquad y = f(x).$$

We have:

First step. $\quad y + \Delta y = f(x + \Delta x).$

Second step. $\quad \Delta y = f(x + \Delta x) - f(x).$

Third step. $\quad \dfrac{\Delta y}{\Delta x} = \dfrac{f(x + \Delta x) - f(x)}{\Delta x}.$

Fourth step. $\quad \dfrac{dy}{dx} = \underset{\Delta x = 0}{\text{limit}} \dfrac{f(x + \Delta x) - f(x)}{\Delta x}.$

The derivative of $f(x)$ is also a function of x. We indicate this result by $f'(x)$, that is, we use the abbreviation

$$(3) \qquad f'(x) = \underset{\Delta x = 0}{\text{limit}} \frac{f(x + \Delta x) - f(x)}{\Delta x}.$$

Summing up, if $y = f(x)$, then

$$(4) \qquad \frac{dy}{dx} = f'(x) = \text{slope of tangent at } (x, y).$$

In words: *the value of the derivative of a function equals the slope of the tangent at the corresponding point on the graph.*

EXAMPLES

1. Differentiate $3x^2 + 5$.

Solution. Applying the successive steps in the *General Rule* we get, after placing

$$y = 3x^2 + 5,$$

First step. $\quad y + \Delta y = 3(x + \Delta x)^2 + 5$

$$= 3x^2 + 6x \cdot \Delta x + 3(\Delta x)^2 + 5.$$

Second step.

$$\begin{array}{l}
y + \Delta y = 3x^2 + 6x \cdot \Delta x + 3(\Delta x)^2 + 5 \\
\underline{y \qquad = 3x^2 \qquad\qquad\qquad\qquad\qquad + 5} \\
\qquad \Delta y = \qquad 6x \cdot \Delta x + 3(\Delta x)^2
\end{array}$$

Third step. $$\frac{\Delta y}{\Delta x} = 6x + 3 \cdot \Delta x.$$

Fourth step. $$\frac{dy}{dx} = 6x. \quad \textit{Ans.}$$

We may also write this

$$\frac{d}{dx}(3x^2 + 5) = 6x.$$

2. Differentiate $x^3 - 2x + 7$.

Solution. Place $y = x^3 - 2x + 7$.

First step.

$$\begin{aligned}
y + \Delta y &= (x + \Delta x)^3 - 2(x + \Delta x) + 7 \\
&= x^3 + 3x^2 \cdot \Delta x + 3x \cdot (\Delta x)^2 + (\Delta x)^3 - 2x - 2 \cdot \Delta x + 7.
\end{aligned}$$

Second step.

$$\begin{array}{l}
y + \Delta y = x^3 + 3x^2 \cdot \Delta x + 3x \cdot (\Delta x)^2 + (\Delta x)^3 - 2x - 2 \cdot \Delta x + 7 \\
\underline{y \qquad = x^3 \qquad\qquad\qquad\qquad\qquad\qquad\qquad - 2x \qquad\qquad + 7} \\
\qquad \Delta y = \qquad 3x^2 \cdot \Delta x + 3x \cdot (\Delta x)^2 + (\Delta x)^3 \qquad - 2 \cdot \Delta x
\end{array}$$

Third step. $$\frac{\Delta y}{\Delta x} = 3x^2 + 3x \cdot \Delta x + (\Delta x)^2 - 2.$$

Fourth step. $$\frac{dy}{dx} = 3x^2 - 2. \quad \textit{Ans.}$$

Or $$\frac{d}{dx}(x^3 - 2x + 7) = 3x^2 - 2.$$

3. Differentiate $\frac{c}{x^2}$.

Solution. Place $y = \frac{c}{x^2}$.

First step. $$y+\Delta y=\frac{c}{(x+\Delta x)^2}.$$

Second step. $$\Delta y=\frac{c}{(x+\Delta x)^2}-\frac{c}{x^2}=\frac{-c\cdot\Delta x(2\,x+\Delta x)}{x^2(x+\Delta x)^2}.$$

Third step. $$\frac{\Delta y}{\Delta x}=-c\cdot\frac{2\,x+\Delta x}{x^2(x+\Delta x)^2}.$$

Fourth step. $$\frac{dy}{dx}=-c\cdot\frac{2\,x}{x^2(x)^2}=-\frac{2\,c}{x^3}.\quad \textit{Ans.}$$

Or $$\frac{d}{dx}\left(\frac{c}{x^2}\right)=\frac{-2\,c}{x^3}.$$

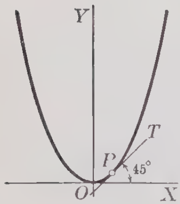
4. Find the slopes of the tangents to the parabola $y=x^2$ at the vertex, and at the point where $x=\frac{1}{2}$.

Solution. Differentiating by the *General Rule*, we get

$$(A)\qquad \frac{dy}{dx}=2\,x=\text{slope of tangent line at } \textit{any} \text{ point on curve.}$$

To find slope of tangent at vertex, substitute $x=0$ in (A),

giving $$\frac{dy}{dx}=0.$$



Therefore the tangent at vertex has the slope zero; that is, it is parallel to the axis of x and in this case coincides with it.

To find slope of tangent at the point P, where $x=\frac{1}{2}$, substituting in (A), giving

$$\frac{dy}{dx}=1;$$

that is, the tangent at P makes an angle of 45° with the x-axis.

The derivative may now be made use of in checking up a plot.

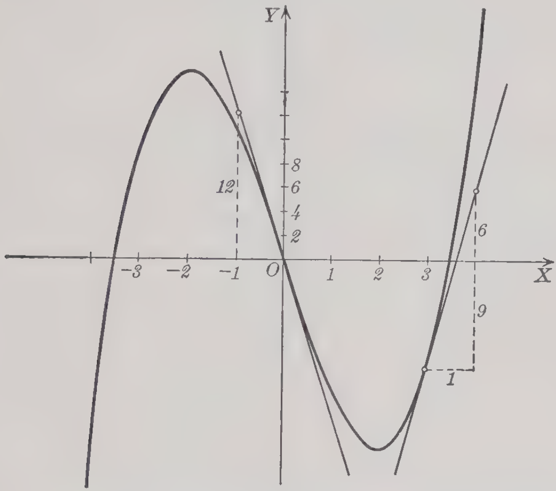
5. Plot the locus of

$$(A)\qquad y=x^3-12\,x,$$

find the slope at each point plotted, and check up in the figure.

Solution. Differentiating by the *General Rule*, we find

$$(B) \qquad \frac{dy}{dx} = 3x^2 - 12 = \text{slope at } (x, y).$$



The table gives the results of the calculation, *e.g.* for $x = 1$, from (A), $y = -11$, and from (B), $\frac{dy}{dx} = 3 - 12 = -9 =$ slope at $(1, -11)$.

x	y	$\frac{dy}{dx}$	x	y	$\frac{dy}{dx}$
0	0	− 12	0	0	− 12
1	− 11	− 9	− 1	11	− 9
2	− 16	0	− 2	16	0
3	− 9	15	− 3	9	15
4	16	36	− 4	− 16	36
etc.	etc.	etc.	etc.	etc.	etc.

The scales used in the figure are those indicated. To construct the tangent at any point, proceed as follows: At (0, 0), the slope $= -12$. Hence lay off from the origin 1 unit to the left and 12 units up. The tangent at (0, 0) passes through this point (p. 18). Similarly, at $(3, -9)$, slope $= 15$, hence lay off 1 unit to the right and 15 units up, the connecting line being the tangent.

Note that *different* scales on x and y *change* the inclination, but the *construction* for the tangent is clear.

Discussion. The graph has a maximum point at $(-2, 16)$ and a minimum at $(2, -16)$. There are *no* other horizontal tangents, since $\frac{dy}{dx} = 3x^2 - 12 = 0$ when $x = \pm 2$, only.

PROBLEMS

Use the *General Rule* in differentiating the following examples.

1. $y = 3x^2$. *Ans.* $\frac{dy}{dx} = 6x$.

2. $y = x^2 - 3x$. *Ans.* $\frac{dy}{dx} = 2x - 3$.

3. $y = ax^2 + bx + c$. *Ans.* $\frac{dy}{dx} = 2ax + b$.

4. $y = x^3$. *Ans.* $\frac{dy}{dx} = 3x^2$.

5. $r = a\theta^2$. *Ans.* $\frac{dr}{d\theta} = 2a\theta$.

6. $p = 2q^2$. *Ans.* $\frac{dp}{dq} = 4q$.

7. $s = t^2 - 2t + 3$. *Ans.* $\frac{ds}{dt} = 2t - 2$.

8. $y = \frac{1}{x}$. *Ans.* $\frac{dy}{dx} = -\frac{1}{x^2}$.

9. $s = \frac{2}{t^2}$. *Ans.* $\frac{ds}{dt} = -\frac{4}{t^3}$.

10. Find the slope of the tangent to the curve $y = 2x^3 - 6x + 5$, (a) at the point where $x = 1$; (b) at the point where $x = 0$.

Ans. (a) 0; (b) -6.

11. (a) Find the slopes of the tangents to the two curves $y = 3x^2 - 1$ and $y = 2x^2 + 3$ at their points of intersection. (b) At what angle do they intersect?

Ans. (a) ± 12, ± 8; (b) arc tan $\frac{4}{97}$.

Plot the locus of each of the following, find the slope at each point plotted, and check up.

12. (a) $y = x^2 - 1$.
(b) $y = 3 - x^2$.
(c) $y = 2 - 4x - x^2$.
(d) $y = 4x + x^2$.
(e) $y = x^2 - 6x$.
(f) $y = x^3 - 3x$.
(g) $y = \frac{1}{3}x^3 - 4x + 1$.
(h) $y = \frac{1}{3}x^3 - x^2 - 3x$.
(i) $y = \frac{1}{3}x^3 - 3x^2 + 5x$.
(j) $y = x^4 - 8x^2 + 10$.

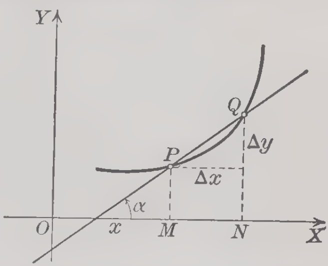
CHAPTER VIII

FORMULAS FOR DIFFERENTIATION

41. Theorems on limits. In the last step of the process of differentiation the increment of the variable is assumed to "vary and approach the limit zero." This statement is made precise by the following definition:

A variable is said to **approach the limit zero** *when its numerical value becomes and remains less than any positive number, however small.*

Again, in differentiation, we start with certain values of x and y, and these values remain *fixed* in the four steps. The actual variables are clearly Δx and Δy. The fact is, of course, that the point $P(x, y)$ is *fixed*, but the point Q moves toward P, and thus Δx and Δy *both* vary. Now in the figure, the position of Q depends only upon the choice of the point N, and the position of N depends upon the value of Δx only. Hence the slope of the secant is a function of Δx only; that is, when P is fixed, we have



(1) $$\tan \alpha = \frac{\Delta y}{\Delta x} = \text{a function of } \Delta x = \phi(\Delta x).$$

The derivative, that is, the slope of the tangent at $P(x, y)$, is the value of

$$\underset{\Delta x = 0}{\text{limit}} \phi(\Delta x);$$

that is, it is the *limiting value of a function of* Δx, *when the variable* Δx *approaches the limit zero.*

It is clear that the successive values of $\phi(\Delta x)$ are the slopes of the successive secants through P and the successive positions of the point Q.

How shall the value of $\underset{\Delta x = 0}{\text{limit}} \phi(\Delta x)$ be found? In all cases thus far the limiting value has been found by substitution directly in the function $\Delta x = 0$. In other words, we have assumed the definition:

The limiting value of a function of Δx *when* Δx *varies and approaches the limit zero is the value of the function when* Δx *equals zero.*

To make it clear that difficulties arise in applying this definition, consider the two examples:

$$\underset{\Delta x = 0}{\text{limit}} \frac{\sqrt{x + \Delta x} - \sqrt{x}}{\Delta x}, \text{ and } \underset{\Delta x = 0}{\text{limit}} \frac{\sin \Delta x}{\Delta x}.$$

Direct substitution leads in each case to the meaningless or indeterminate expression $\frac{0}{0}$. In each of these examples, therefore, the definition does not apply, and other methods must be used (see Arts. 51 and 54).

In the following pages, repeated application is made of the following theorems, whose proof is here omitted.

Given a number of variables whose limits are known; then

I. *The limit of an algebraic sum of any number of variables equals the same algebraic sum of their respective limits.*

II. *The limit of the product of any number of variables equals the product of their respective limits.*

III. *The limit of a quotient of two variables equals the quotient of their respective limits when the limit of the denominator is not zero.*

42. Fundamental formulas. For economy of time in differentiation, special rules have been devised, which are here set down, the proofs being given in later sections.

In each formula, u, v, w, etc., are assumed to be functions of the same variable x, and a, c, e, and n are constants.

I $$\frac{dc}{dx} = 0.$$

II $$\frac{dx}{dx} = 1.$$

III $$\frac{d}{dx}(u + v - w) = \frac{du}{dx} + \frac{dv}{dx} - \frac{dw}{dx}.$$

IV $$\frac{d}{dx}(cv) = c\frac{dv}{dx}.$$

V $$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}.$$

VI $$\frac{d}{dx}(v^n) = nv^{n-1}\frac{dv}{dx}.$$

VI *a* $$\frac{d}{dx}(x^n) = nx^{n-1}.$$

VII $$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}.$$

VII *a* $$\frac{d}{dx}\left(\frac{u}{c}\right) = \frac{\dfrac{du}{dx}}{c}.$$

VIII $$\frac{d}{dx}(\log_a v) = \frac{\log_a e}{v}\frac{dv}{dx}.$$

VIII *a* $$\frac{d}{dx}(\log v) = \frac{1}{v}\frac{dv}{dx}.$$

IX $$\frac{d}{dx}(a^v) = a^v \log a\frac{dv}{dx}.$$

IX *a* $$\frac{d}{dx}(e^v) = e^v\frac{dv}{dx}.$$

X $$\frac{d}{dx}(\sin v) = \cos v\frac{dv}{dx}.$$

XI $$\frac{d}{dx}(\cos v) = -\sin v \frac{dv}{dx}.$$

XII $$\frac{d}{dx}(\tan v) = \sec^2 v \frac{dv}{dx}.$$

XIII $$\frac{d}{dx}(\cot v) = -\csc^2 v \frac{dv}{dx}.$$

XIV $$\frac{d}{dx}(\sec v) = \sec v \tan v \frac{dv}{dx}.$$

XV $$\frac{d}{dx}(\csc v) = -\csc v \cot v \frac{dv}{dx}.$$

XVI $$\frac{d}{dx}(\text{arc} \sin v) = \frac{1}{\sqrt{1 - v^2}} \frac{dv}{dx}.$$

XVII $$\frac{d}{dx}(\text{arc} \tan v) = \frac{1}{1 + v^2} \frac{dv}{dx}.$$

43. Differentiation of a constant. A function that is known to have the same value for every value of the independent variable is constant, and we may denote it by

$$y = c.$$

As x takes on an increment Δx, the function does not change in value; that is, $\Delta y = 0$, and

$$\frac{\Delta y}{\Delta x} = 0.$$

But $$\underset{\Delta x = 0}{\text{limit}} \left(\frac{\Delta y}{\Delta x}\right) = \frac{dy}{dx} = 0.$$

I $$\therefore \frac{dc}{dx} = 0.$$

The derivative of a constant is zero.

44. Differentiation of a variable with respect to itself.

Let $$y = x.$$

Following the *General Rule*, p. 104, we have

First step. $$y + \Delta y = x + \Delta x.$$

Second step. $$\Delta y = \Delta x.$$

Third step. $$\frac{\Delta y}{\Delta x} = 1.$$

Fourth step. $$\frac{dy}{dx} = 1.$$

II $$\therefore \frac{dx}{dx} = 1.$$

The derivative of a variable with respect to itself is unity.

45. Differentiation of a sum.

Let $$y = u + v - w.$$

By the *General Rule: First step.* Changing x to $x + \Delta x$,

$$y + \Delta y = u + \Delta u + v + \Delta v - w - \Delta w.$$

Second step. $$\Delta y = \Delta u + \Delta v - \Delta w.$$

Third step. $$\frac{\Delta y}{\Delta x} = \frac{\Delta u}{\Delta x} + \frac{\Delta v}{\Delta x} - \frac{\Delta w}{\Delta x}.$$

Fourth step. $$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} - \frac{dw}{dx}.$$

[Applying Theorem I, p. 112.]

III $$\therefore \frac{d}{dx}(u + v - w) = \frac{du}{dx} + \frac{dv}{dx} - \frac{dw}{dx}.$$

Similarly for the algebraic sum of any finite number of functions.

The derivative of the algebraic sum of a finite number of functions is equal to the same algebraic sum of their derivatives.

46 Differentiation of the product of a constant and a function.

Let $$y = cv.$$

First step. Changing x to $x + \Delta x$,

$$y + \Delta y = c(v + \Delta v) = cv + c\Delta v.$$

Second step. $$\Delta y = c \cdot \Delta v.$$

Third step. $$\frac{\Delta y}{\Delta x} = c\frac{\Delta v}{\Delta x}.$$

Fourth step. $$\frac{dy}{dx} = c\frac{dv}{dx}.$$

[Applying Theorem II, p. 112.]

IV $$\therefore \frac{d}{dx}(cv) = c\frac{dv}{dx}.$$

The derivative of the product of a constant and a function is equal to the product of the constant and the derivative of the function.

47. Differentiation of the product of two functions.

Let $$y = uv.$$

First step. Changing x to $x + \Delta x$,

$$\begin{aligned} y + \Delta y &= (u + \Delta u)(v + \Delta v) \\ &= uv + u \cdot \Delta v + v \cdot \Delta u + \Delta u \cdot \Delta v. \end{aligned}$$

Second step. $$\Delta y = u \cdot \Delta v + v \cdot \Delta u + \Delta u \cdot \Delta v.$$

Third step. $$\frac{\Delta y}{\Delta x} = u\frac{\Delta v}{\Delta x} + v\frac{\Delta u}{\Delta x} + \Delta u\frac{\Delta v}{\Delta x}.$$

Fourth step. $$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}.$$

[Applying Theorem II, p. 112, since when Δx approaches zero as a limit, Δu also approaches zero as a limit, and limit $\left(\Delta u\frac{\Delta v}{\Delta x}\right) = 0.$]

V $$\therefore \frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}.$$

The derivative of the product of two functions is equal to the first function times the derivative of the second, plus the second function times the derivative of the first.

To extend V to the product of three functions, proceed thus:

$$\frac{d}{dx}(uvw) = \frac{d}{dx}(uv \cdot w),$$

where we now regard uv as one function.

(1) $$\therefore \frac{d}{dx}(uv \cdot w) = uv\frac{dw}{dx} + w\frac{d}{dx}(uv)$$

$$= uv\frac{dw}{dx} + wu\frac{dv}{dx} + wv\frac{du}{dx}.$$

The general rule to be read out of this result is:

The derivative of the product of any number of functions equals the sum of all the products that can be formed by multiplying the derivative of one function by all the remaining functions.

48. Differentiation of a power of a function. In equation (1) of Art. 47, if we assume u and w to be identical with v, we obtain at once

$$\frac{d}{dx}(v^3) = 3\,v^2\frac{dv}{dx},$$

which proves formula VI for $n = 3$. In a similar manner we may prove VI for any positive integer. For the present we assume VI to hold if n is any constant, proof being reserved for a later section.

In VI, putting $v = x$, we obtain VI a, using II.

Power rule. *The derivative of a function with a constant exponent equals the product of the exponent, the function with exponent less one, and the derivative of the function.*

49. Differentiation of a quotient.

Let $$y = \frac{u}{v},\ v \neq 0.$$

First step. Changing x to $x + \Delta x$,

$$y + \Delta y = \frac{u + \Delta u}{v + \Delta v}.$$

Second step. $$\Delta y = \frac{u + \Delta u}{v + \Delta v} - \frac{u}{v} = \frac{v \cdot \Delta u - u \cdot \Delta v}{v(v + \Delta v)}.$$

Third step. $$\frac{\Delta y}{\Delta x} = \frac{v \dfrac{\Delta u}{\Delta x} - u \dfrac{\Delta v}{\Delta x}}{v(v + \Delta v)}.$$

Fourth step. $$\frac{dy}{dx} = \frac{v \dfrac{du}{dx} - u \dfrac{dv}{dx}}{v^2}.$$

[Applying Theorems II and III, p. 112.]

VII $$\therefore \frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v \dfrac{du}{dx} - u \dfrac{dv}{dx}}{v^2}.$$

The derivative of a fraction is equal to the denominator times the derivative of the numerator, minus the numerator times the derivative of the denominator, all divided by the square of the denominator.

When the denominator is constant, set $v = c$ in VII, giving

VII *a* $$\frac{d}{dx}\left(\frac{u}{c}\right) = \frac{\dfrac{du}{dx}}{c}.$$

$$\left[\text{Since } \frac{dv}{dx} = \frac{dc}{dx} = 0.\right]$$

We may also get VII a from IV as follows:

$$\frac{d}{dx}\left(\frac{u}{c}\right) = \frac{1}{c}\frac{du}{dx} = \frac{\dfrac{du}{dx}}{c}.$$

The derivative of the quotient of a function by a constant is equal to the derivative of the function divided by the constant.

PROBLEMS

Differentiate the following:

1. $y = x^3$.

Solution. $\frac{dy}{dx}=\frac{d}{dx}(x^3)=3\,x^2.$ *Ans.* (by VI a)

$[n=3.]$

2. $y=ax^4-bx^2.$

Solution. $\frac{dy}{dx}=\frac{d}{dx}(ax^4-bx^2)=\frac{d}{dx}(ax^4)-\frac{d}{dx}(bx^2)$ (by III)

$=a\frac{d}{dx}(x^4)-b\frac{d}{dx}(x^2)$ (by IV)

$=4\,ax^3-2\,bx.$ *Ans.* (by VI a)

3. $y=x^{\frac{4}{3}}+5.$

Solution. $\frac{dy}{dx}=\frac{d}{dx}(x^{\frac{4}{3}})+\frac{d}{dx}(5)$ (by III)

$=\frac{4}{3}\,x^{\frac{1}{3}}.$ *Ans.* (by VI a and I)

4. $y=\frac{3\,x^3}{\sqrt[5]{x^2}}-\frac{7\,x}{\sqrt[3]{x^4}}+8\sqrt[7]{x^3}.$

Solution. $\frac{dy}{dx}=\frac{d}{dx}(3\,x^{\frac{13}{5}})-\frac{d}{dx}(7x^{-\frac{1}{3}})+\frac{d}{dx}(8\,x^{\frac{3}{7}})$ (by III)

$=\frac{39}{5}\,x^{\frac{8}{5}}+\frac{7}{3}\,x^{-\frac{4}{3}}+\frac{24}{7}\,x^{-\frac{4}{7}}.$

Ans. (by IV and VI a)

5. $y=(x^2-3)^5.$

Solution. $\frac{dy}{dx}=5(x^2-3)^4\frac{d}{dx}(x^2-3)$ (by VI)

$[v=x^2-3 \text{ and } n=4.]$

$=5(x^2-3)^4\cdot 2\,x=10\,x(x^2-3)^4.$ *Ans.*

We might have expanded this function by the Binomial Theorem and then applied III, etc., but the above process is to be preferred.

6. $y=\sqrt{a^2-x^2}.$

Solution. $\frac{dy}{dx}=\frac{d}{dx}(a^2-x^2)^{\frac{1}{2}}=\frac{1}{2}(a^2-x^2)^{-\frac{1}{2}}\frac{d}{dx}(a^2-x^2)$ (by VI)

$[v=a^2-x^2 \text{ and } n=\frac{1}{2}.]$

$$= \tfrac{1}{2}(a^2 - x^2)^{-\frac{1}{2}}(-2x) = -\frac{x}{\sqrt{a^2 - x^2}}. \quad \textit{Ans.}$$

7. $y = (3x^2 + 2)\sqrt{1 + 5x^2}.$

Solution. $$\frac{dy}{dx} = (3x^2 + 2)\frac{d}{dx}(1 + 5x^2)^{\frac{1}{2}} + (1 + 5x^2)^{\frac{1}{2}}\frac{d}{dx}(3x^2 + 2)$$

(by V)

$[u = 3x^2 + 2 \text{ and } v = (1 + 5x^2)^{\frac{1}{2}}.]$

$$= (3x^2 + 2)\tfrac{1}{2}(1 + 5x^2)^{-\frac{1}{2}}\frac{d}{dx}(1 + 5x^2) + (1 + 5x^2)^{\frac{1}{2}}6x$$

(by VI, etc.)

$$= (3x^2 + 2)(1 + 5x^2)^{-\frac{1}{2}}5x + 6x(1 + 5x^2)^{\frac{1}{2}}$$

$$= \frac{5x(3x^2 + 2)}{\sqrt{1 + 5x^2}} + 6x\sqrt{1 + 5x^2} = \frac{45x^3 + 16x}{\sqrt{1 + 5x^2}}.$$

Ans.

8. $y = \dfrac{a^2 + x^2}{\sqrt{a^2 - x^2}}.$

Solution. $$\frac{dy}{dx} = \frac{(a^2 - x^2)^{\frac{1}{2}}\frac{d}{dx}(a^2 + x^2) - (a^2 + x^2)\frac{d}{dx}(a^2 - x^2)^{\frac{1}{2}}}{a^2 - x^2}$$

(by VII)

$$= \frac{2x(a^2 - x^2) + x(a^2 + x^2)}{(a^2 - x^2)^{\frac{3}{2}}}$$

[Multiplying both numerator and denominator by $(a^2 - x^2)^{\frac{1}{2}}$.]

$$= \frac{3a^2x - x^3}{(a^2 - x^2)^{\frac{3}{2}}}. \quad \textit{Ans.}$$

9. $y = 5x^4 + 3x^2 - 6.$ $\qquad \dfrac{dy}{dx} = 20x^3 + 6x.$

10. $y = 3cx^2 - 8dx + 5e.$ $\qquad \dfrac{dy}{dx} = 6cx - 8d.$

11. $y = x^{a+b}.$ $\qquad \dfrac{dy}{dx} = (a + b)x^{a+b-1}.$

12. $y = x^n + nx + n.$ $\qquad \dfrac{dy}{dx} = nx^{n-1} + n.$

13. $f(x) = \frac{2}{3}x^3 - \frac{3}{2}x^2 + 5.$ $\qquad f'(x) = 2x^2 - 3x.$

14. $f(x) = (a+b)x^2 + cx + d.$ $\qquad f'(x) = 2(a+b)x + c.$

15. $\dfrac{d}{dx}(a + bx + cx^2) = b + 2cx.$

16. $\dfrac{d}{dy}(5y^m - 3y + 6) = 5my^{m-1} - 3.$

17. $v = \dfrac{v_0 p_0}{p}.$ $\qquad \dfrac{dv}{dp} = -\dfrac{v_0 p_0}{p^2}.$

18. $v = v_0 + ft.$ $\qquad \dfrac{dv}{dt} = f.$

19. $s = s_0 + v_0 t + \frac{1}{2}ft^2.$ $\qquad \dfrac{ds}{dt} = v_0 + ft.$

20. $l = 1 + b\theta + c\theta^2.$ $\qquad \dfrac{dl}{d\theta} = b + 2c\theta.$

21. $s = 2t^2 + 3t + 5.$ $\qquad \dfrac{ds}{dt} = 4t + 3.$

22. $s = at^3 - bt^2 + c.$ $\qquad \dfrac{ds}{dt} = 3at^2 - 2bt.$

23. $r = a\theta^2.$ $\qquad \dfrac{dr}{d\theta} = 2a\theta.$

24. $r = c\theta^3 + d\theta^2 + e\theta.$ $\qquad \dfrac{dr}{d\theta} = 3c\theta^2 + 2d\theta + e.$

25. $y = 6x^{\frac{7}{2}} + 4x^{\frac{5}{2}} + 2x^{\frac{3}{2}}.$ $\qquad \dfrac{dy}{dx} = 21x^{\frac{5}{2}} + 10x^{\frac{3}{2}} + 3x^{\frac{1}{2}}.$

26. $y = \sqrt{3x} + \sqrt[3]{x} + \dfrac{1}{x}.$ $\qquad \dfrac{dy}{dx} = \dfrac{3}{2\sqrt{3x}} + \dfrac{1}{3\sqrt[3]{x^2}} - \dfrac{1}{x^2}.$

27. $y = \dfrac{a + bx + cx^2}{x}.$ $\qquad \dfrac{dy}{dx} = c - \dfrac{a}{x^2}.$

28. $y = \dfrac{(x-1)^3}{x^{\frac{1}{3}}}$. $\qquad \dfrac{dy}{dx} = \frac{8}{3}x^{\frac{5}{3}} - 5x^{\frac{2}{3}} + 2x^{-\frac{1}{3}} + \frac{1}{3}x^{-\frac{4}{3}}$.

29. $y = \dfrac{x^{\frac{5}{2}} - x - x^{\frac{1}{2}} + a}{x^{\frac{3}{2}}}$. $\qquad \dfrac{dy}{dx} = \dfrac{2x^{\frac{5}{2}} + x + 2x^{\frac{1}{2}} - 3a}{2x^{\frac{5}{2}}}$.

30. $y = (2x^3 + x^2 - 5)^3$. $\qquad \dfrac{dy}{dx} = 6x(3x+1)(2x^3+x^2-5)^2$.

31. $f(x) = (a + bx^2)^{\frac{5}{4}}$. $\qquad f'(x) = \dfrac{5bx}{2}(a+bx^2)^{\frac{1}{4}}$.

32. $f(x) = (1+4x^3)(1+2x^2)$. $\qquad f'(x) = 4x(1+3x+10x^3)$.

33. $f(x) = (a+x)\sqrt{a-x}$. $\qquad f'(x) = \dfrac{a-3x}{2\sqrt{a-x}}$.

34. $f(x) = (a+x)^m(b+x)^n$. $\qquad f'(x) = (a+x)^m(b+x)^n\left[\dfrac{m}{a+x} + \dfrac{n}{b+x}\right]$.

35. $y = \dfrac{1}{x^n}$. $\qquad \dfrac{dy}{dx} = -\dfrac{n}{x^{n+1}}$.

36. $y = x(a^2+x^2)\sqrt{a^2-x^2}$. $\qquad \dfrac{dy}{dx} = \dfrac{a^4 + a^2x^2 - 4x^4}{\sqrt{a^2-x^2}}$.

37. $y = \dfrac{2x^4}{b^2-x^2}$. $\qquad \dfrac{dy}{dx} = \dfrac{8b^2x^3 - 4x^5}{(b^2-x^2)^2}$.

38. $y = \dfrac{a-x}{a+x}$. $\qquad \dfrac{dy}{dx} = -\dfrac{2a}{(a+x)^2}$.

39. $s = \dfrac{t^3}{(1+t)^2}$. $\qquad \dfrac{ds}{dt} = \dfrac{3t^2+t^3}{(1+t)^3}$.

40. $f(s) = \dfrac{(s+4)^2}{s+3}$. $\qquad f'(s) = \dfrac{(s+2)(s+4)}{(s+3)^2}$.

41. $f(\theta) = \dfrac{\theta}{\sqrt{a-b\theta^2}}$. $\qquad f'(\theta) = \dfrac{a}{(a-b\theta^2)^{\frac{3}{2}}}$.

42. $F(r) = \sqrt{\dfrac{1+r}{1-r}}$. $\qquad F'(r) = \dfrac{1}{(1-r)\sqrt{1-r^2}}$.

43. $\psi(y)=\left(\frac{y}{1-y}\right)^m.$ $\qquad\psi'(y)=\frac{my^{m-1}}{(1-y)^{m+1}}.$

44. $\phi(x)=\frac{2x^2-1}{x\sqrt{1+x^2}}.$ $\qquad\phi'(x)=\frac{1+4x^2}{x^2(1+x^2)^{\frac{3}{2}}}.$

45. $\frac{d}{dx}\left[\frac{1}{(a+x)^m(b+x)^n}\right]=-\frac{m(b+x)+n(a+x)}{(a+x)^{m+1}(b+x)^{n+1}}.$

46. $\frac{d}{dx}\left[\frac{\sqrt{a+x}+\sqrt{a-x}}{\sqrt{a+x}-\sqrt{a-x}}\right]=-\frac{a^2+a\sqrt{a^2-x^2}}{x^2\sqrt{a^2-x^2}}.$

Hint. Rationalize the denominator first.

47. $y=\sqrt{2px}.$ $\qquad\frac{dy}{dx}=\frac{p}{y}.$

48. $y=\frac{b}{a}\sqrt{a^2-x^2}.$ $\qquad\frac{dy}{dx}=-\frac{b^2x}{a^2y}.$

49. $y=(a^{\frac{2}{3}}-x^{\frac{2}{3}})^{\frac{3}{2}}.$ $\qquad\frac{dy}{dx}=-\sqrt[3]{\frac{y}{x}}.$

50. $r=\sqrt{a\phi}+c\sqrt{\phi^3}.$ $\qquad\frac{dr}{d\phi}=\frac{\sqrt{a}+3c\phi}{2\sqrt{\phi}}.$

51. $u=\frac{v^c+v^d}{cd}.$ $\qquad\frac{du}{dv}=\frac{v^{c-1}}{d}+\frac{v^{d-1}}{c}.$

52. $p=\frac{(q+1)^{\frac{3}{2}}}{\sqrt{q-1}}.$ $\qquad\frac{dp}{dq}=\frac{(q-2)\sqrt{q+1}}{(q-1)^{\frac{3}{2}}}.$

50. Differentiation of a function of a function. It sometimes happens that y, instead of being defined directly as a function of x, is given as a function of another variable v which is defined as a function of x. In that case y is a function of x through v and is called a *function of a function.*

For example, if
$$y=\frac{2v}{1-v^2},$$
and
$$v=1-x^2,$$

then y is a function of a function. By eliminating v we may express y directly as a function of x, but in general this is not the best plan when we wish to find $\frac{dy}{dx}$.

Given then

$$y=f(v),\ v=\phi(x), \text{ and } \therefore\ y=F(x).$$

To differentiate, changing x to $x+\Delta x$, then v and y become $v+\Delta v$ and $y+\Delta y$ respectively. The third step will give the values of

$$\frac{\Delta y}{\Delta v},\ \frac{\Delta v}{\Delta x}, \text{ and } \frac{\Delta y}{\Delta x}.$$

But by direct multiplication

$$\frac{\Delta y}{\Delta v}\cdot\frac{\Delta v}{\Delta x}=\frac{\Delta y}{\Delta x}.$$

Taking the fourth step, we obtain, applying Theorem II, p. 112, after interchanging the members,

(*A*) $$\frac{dy}{dx}=\frac{dy}{dv}\cdot\frac{dv}{dx}.$$

If $y=f(v)$ and $v=\phi(x)$, the derivative of y with respect to x equals the product of the derivative of y with respect to v and the derivative of v with respect to x.

51. Differentiation of a logarithm. It is required to derive a formula for

(1) $$\frac{d}{dx}\log_a v,$$

when v is a function of x, and the logarithm is taken in any system whose base is a positive constant a. Let us proceed on the basis of Art. 50, taking at first v for the variable.

Let $$y=\log_a v.$$

First step. Changing v to $v+\Delta v$, then

$$y+\Delta y=\log_a(v+\Delta v).$$

Second step. $\Delta y = \log_a (v + \Delta v) - \log_a v.$

Third step. $\dfrac{\Delta y}{\Delta v} = \dfrac{\log_a (v + \Delta v) - \log_a v}{\Delta v}.$

Fourth step. $\dfrac{dy}{dv} = \dfrac{0}{0};$

that is, the limiting value of the right-hand member in the third step cannot be found by *direct* substitution (Art. 41, p. 112). We therefore examine the result of the second step,

(2) $$\Delta y = \log_a (v + \Delta v) - \log_a v,$$

and endeavor to transform the right-hand member. By 15, p. 1, we may write (2) in the form,

(3) $$\Delta y = \log_a \left(\frac{v + \Delta v}{v}\right) = \log_a \left(1 + \frac{\Delta v}{v}\right).$$

Dividing by Δv, then

(4) $$\frac{\Delta y}{\Delta v} = \frac{1}{\Delta v} \log_a \left(1 + \frac{\Delta v}{v}\right).$$

We observe that the fraction $\dfrac{\Delta v}{v}$ occurs in the logarithm. We may introduce this fraction also before the logarithm by multiplying and dividing by v. This gives

(5) $$\frac{\Delta y}{\Delta v} = \frac{1}{v} \cdot \frac{v}{\Delta v} \log_a \left(1 + \frac{\Delta v}{v}\right).$$

The factor $\dfrac{v}{\Delta v}$ in front of the logarithm may be written as an exponent on the parenthesis (16, p. 1), and hence

(6) $$\frac{\Delta y}{\Delta v} = \frac{1}{v} \log_a \left[\left(1 + \frac{\Delta v}{v}\right)^{\frac{v}{\Delta v}}\right].$$

Consider now the expression within the square brackets. If we set $z = \dfrac{\Delta v}{v}$, this will have the form

(7) $$(1 + z)^{\frac{1}{z}}.$$

The fourth step requires the value of

(8) $$\underset{z=0}{\text{limit}}(1+z)^{\frac{1}{z}},$$

since obviously, when $\Delta v = 0$, also $z = \dfrac{\Delta v}{v} = 0$, the value of v remaining constant in differentiation (Art. 41). Direct substitution of $z = 0$ in (7) gives, however, the meaningless expression 1^∞. It then becomes necessary to assume values of z as close to zero as we choose and calculate the value of (7). The limiting value thus obtained, for example, by setting

$$z = 1,\ \tfrac{1}{10},\ \tfrac{1}{100},\ \tfrac{1}{1000},\ \text{etc.},$$

is the number e (Art. 33, p. 79), the natural base; that is,* *as a definition*, we set

(9) $$e = \underset{z=0}{\text{limit}}(1+z)^{\frac{1}{z}}.$$

The calculation of e to two decimal places is easy. For example, using a seven-place table of common logarithms, we find the values

$z = 1$	$\frac{1}{10}$	$\frac{1}{100}$	$\frac{1}{1000}$
$(1+z)^{\frac{1}{z}} = 2$	2.59	2.70	2.71

Returning now to (6) and letting Δv approach the limit zero, we obtain the required result

(10) $$\frac{dy}{dv} = \frac{1}{v}\log_a e.$$

Since v is a function of x and it is required to differentiate $\log_a v$ with respect to x, we must use formula (A), Art. 50, for differentiating a function of a function; namely,

$$\frac{dy}{dx} = \frac{dy}{dv} \cdot \frac{dv}{dx}.$$

* The number π ($= 3.1416$), with which the student is familiar in geometry, is defined also as a limit: namely, in a fixed circle, π is the limiting value of the ratio of the perimeter of an inscribed regular polygon to the diameter, when the number of sides increases indefinitely.

Substituting the value of $\frac{dy}{dv}$ from (10), we get, since $y = \log_a v$,

VIII $$\frac{d}{dx}(\log_a v) = \frac{\log_a e}{v}\frac{dv}{dx}.$$

When $a = e$ this becomes

VIII *a* $$\frac{d}{dx}(\log v) = \frac{1}{v}\frac{dv}{dx},$$

since $$\log_e e = 1 \text{ (19, p. 1).}$$

In VIII *a*, the natural base is omitted in writing down $\log v$; that is, when no base is indicated it is assumed that natural logarithms are used.

Putting $a = 10$, VIII becomes

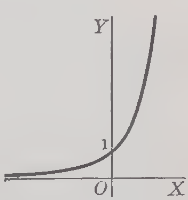
(11) $$\frac{d}{dx}\log_{10} v = \frac{\log_{10} e}{v}\frac{dv}{dx} = \frac{M}{v}\frac{dv}{dx},$$

where M (Art. 33, (6)) is the modulus of the common system. Formula VIII *a* justifies the introduction of the number e, for in the Calculus the use of natural logarithms renders unnecessary writing down M, and this results in a great saving of labor.

The derivative of the logarithm of a function equals the product of the modulus of the system, the reciprocal of the function, and the derivative of the function.

52. Differentiation of an exponential function. Let us find the slope at any point (x, y) on the exponential curve (Art. 33)

(1) $$y = e^x.$$



Using natural logarithms, then

(2) $$x = \log y.$$

Differentiating with respect to y,

(3) $$\frac{dx}{dy} = \frac{1}{y}, \text{ by VIII } a.$$

We wish the value of $\frac{dy}{dx}$. Now if we had

differentiated (1) by the *General Rule* the third step would have given the value of the ratio $\frac{\Delta y}{\Delta x}$. Similarly, from (2) we should have found the value of $\frac{\Delta x}{\Delta y}$. But by direct multiplication,

$$\frac{\Delta y}{\Delta x} \cdot \frac{\Delta x}{\Delta y} = 1. \tag{4}$$

Now take the fourth step. Then we have at once

$$\frac{dy}{dx} \cdot \frac{dx}{dy} = 1,$$

or the useful formula

$$\boldsymbol{\frac{dx}{dy} = 1 \div \frac{dy}{dx}}. \tag{5}$$

When we differentiate (1), we wish the value of $\frac{dy}{dx}$. From (5), and (3),

$$\frac{dy}{dx} = 1 \div \frac{dx}{dy} = y. \tag{6}$$

Referring to (1), we have the formula

$$\frac{d}{dx}(e^x) = e^x. \tag{7}$$

In other words, the exponential function e^x possesses the remarkable property of being its own derivative. Changing x to v in (7) gives

$$\frac{d}{dv} e^v = e^v. \tag{8}$$

From this result and (*A*), Art. 50, we derive

IX *a*
$$\boldsymbol{\frac{d}{dx} e^v = e^v \frac{dv}{dx}}.$$

The derivative of the natural base with a function as exponent equals the whole expression times the derivative of the exponent.

To derive IX, make use of the equation

(9) $$e^{\log a} = a,$$

which follows at once from the definition of a logarithm (Art. 33, (1)). Hence

(10) $$a^v = e^{v \log a}.$$

$$\therefore \frac{d}{dx} a^v = \frac{d}{dx} e^{v \log a} = e^{v \log a} \frac{d}{dx}(v \log a) = e^{v \log a} \log a \frac{dv}{dx}.$$

Hence, substituting from (10), we obtain

IX $$\boldsymbol{\frac{d}{dx} a^v = a^v \log a \frac{dv}{dx}.}$$

53. Proof of the power rule. We may now complete the proof of **VI** postponed from Art. 48. Since, as in (9) of the preceding section,

$$e^{\log v} = v,$$

then

(1) $$v^n = e^{n \log v}.$$

$$\therefore \frac{d}{dx} v^n = \frac{d}{dx} e^{n \log v} = e^{n \log v} \frac{d}{dx}(n \log v) = e^{n \log v} \frac{n}{v} \frac{dv}{dx}.$$

Substituting from (1), this becomes

$$\frac{d}{dx} v^n = \frac{nv^n}{v} \frac{dv}{dx} = nv^{n-1} \frac{dv}{dx},$$

which establishes the formula.

EXAMPLES

1. Differentiate $y = \log \sqrt{1 - x^2}$.

Solution. $$\frac{dy}{dx} = \frac{\frac{d}{dx}(1 - x^2)^{\frac{1}{2}}}{(1 - x^2)^{\frac{1}{2}}} \qquad \text{(by VIII } a\text{)}$$

$$= \frac{\frac{1}{2}(1 - x^2)^{-\frac{1}{2}}(-2x)}{(1 - x^2)^{\frac{1}{2}}} \qquad \text{(by VI)}$$

$$= \frac{x}{x^2 - 1}. \quad Ans.$$

2. Differentiate $y = a^{3x^2}$.

Solution.
$$\frac{dy}{dx} = \log a \cdot a^{3x^2} \frac{d}{dx}(3x^2). \qquad \text{(by IX)}$$
$$= 6x \log a \cdot a^{3x^2}. \quad \textit{Ans.}$$

3. Differentiate $y = be^{c^2+x^2}$.

Solution.
$$\frac{dy}{dx} = b \frac{d}{dx}(e^{c^2+x^2}) \qquad \text{(by IV)}$$
$$= be^{c^2+x^2} \frac{d}{dx}(c^2 + x^2) \qquad \text{(by IX } a\text{)}$$
$$= 2bxe^{c^2+x^2}. \quad \textit{Ans.}$$

4. Differentiate $y = \log \sqrt{\dfrac{1+x^2}{1-x^2}}$.

Solution. Simplifying by means of 17 and 15, p. 1,
$$y = \tfrac{1}{2}[\log(1+x^2) - \log(1-x^2)].$$
$$\frac{dy}{dx} = \frac{1}{2}\left[\frac{\frac{d}{dx}(1+x^2)}{1+x^2} - \frac{\frac{d}{dx}(1-x^2)}{1-x^2}\right] \qquad \text{(by VIII } a\text{, etc.)}$$
$$= \frac{x}{1+x^2} + \frac{x}{1-x^2} = \frac{2x}{1-x^4}. \quad \textit{Ans.}$$

PROBLEMS

Differentiate the following:

1. $y = \log(x + a)$. $\qquad \dfrac{dy}{dx} = \dfrac{1}{x+a}$.

2. $y = \log(ax + b)$. $\qquad \dfrac{dy}{dx} = \dfrac{a}{ax+b}$.

3. $y = \log \dfrac{1+x}{1-x}$. $\qquad \dfrac{dy}{dx} = \dfrac{2}{1-x^2}$.

4. $y = \log \dfrac{1+x^2}{1-x^2}$. $\qquad \dfrac{dy}{dx} = \dfrac{4x}{1-x^4}$.

5. $y = e^{ax}$. $\qquad \dfrac{dy}{dx} = ae^{ax}$.

6. $y = e^{4x+5}$. $\qquad \dfrac{dy}{dx} = 4\,e^{4x+5}$.

7. $y = \log(x^2 + x)$. $\qquad \dfrac{dy}{dx} = \dfrac{2\,x+1}{x^2+x}$.

8. $y = \log(x^3 - 2\,x + 5)$. $\qquad \dfrac{dy}{dx} = \dfrac{3\,x^2 - 2}{x^3 - 2\,x + 5}$.

9. $y = \log_a(2\,x + x^3)$. $\qquad \dfrac{dy}{dx} = \log_a e \cdot \dfrac{2 + 3\,x^2}{2\,x + x^3}$.

10. $y = x \log x$. $\qquad \dfrac{dy}{dx} = \log x + 1$.

11. $f(x) = \log x^3$. $\qquad f'(x) = \dfrac{3}{x}$.

12. $f(x) = \log^3 x$. $\qquad f'(x) = \dfrac{3 \log^2 x}{x}$.

Hint. $\log^3 x = (\log x)^3$. Use first VI, $v = \log x$, $n = 3$; and then VIII.

13. $f(x) = \log \dfrac{a+x}{a-x}$. $\qquad f'(x) = \dfrac{2\,a}{a^2 - x^2}$.

14. $f(x) = \log(x + \sqrt{1+x^2})$. $\qquad f'(x) = \dfrac{1}{\sqrt{1+x^2}}$.

15. $y = a^{e^x}$. $\qquad \dfrac{dy}{dx} = \log a \cdot a^{e^x} e^x$.

16. $y = b^{x^2}$. $\qquad \dfrac{dy}{dx} = 2\,x \log b \cdot b^{x^2}$.

17. $y = 7^{x^2+2x}$. $\qquad \dfrac{dy}{dx} = 2 \log 7 \cdot (x+1) 7^{x^2+2x}$.

18. $y = c^{a^2-x^2}$. $\qquad \dfrac{dy}{dx} = -\,2\,x \log c \cdot c^{a^2-x^2}$.

19. $r = a^\theta$. $\qquad \dfrac{dr}{d\theta} = a^\theta \log a.$

20. $r = a^{\log\theta}$. $\qquad \dfrac{dr}{d\theta} = \dfrac{a^{\log\theta}\log a}{\theta}.$

21. $s = e^{b^2+t^2}$. $\qquad \dfrac{ds}{dt} = 2\,te^{b^2+t^2}.$

22. $u = ae^{\sqrt{v}}$. $\qquad \dfrac{du}{dv} = \dfrac{ae^{\sqrt{v}}}{2\sqrt{v}}.$

23. $p = e^{q\log q}$. $\qquad \dfrac{dp}{dq} = e^{q\log q}(1+\log q).$

24. $\dfrac{d}{dx}[e^x(1-x^2)] = e^x(1-2\,x-x^2).$

25. $\dfrac{d}{dx}\left(\dfrac{e^x-1}{e^x+1}\right) = \dfrac{2\,e^x}{(e^x+1)^2}.$

26. $\dfrac{d}{dx}(x^2e^{ax}) = xe^{ax}(ax+2).$

27. $y = \log\dfrac{e^x}{1+e^x}$. $\qquad \dfrac{dy}{dx} = \dfrac{1}{1+e^x}.$

28. $y = \dfrac{a}{2}(e^{\frac{x}{a}} - e^{-\frac{x}{a}})$. $\qquad \dfrac{dy}{dx} = \dfrac{1}{2}(e^{\frac{x}{a}} + e^{-\frac{x}{a}}).$

29. $y = \dfrac{e^x-e^{-x}}{e^x+e^{-x}}$. $\qquad \dfrac{dy}{dx} = \dfrac{4}{(e^x+e^{-x})^2}.$

30. $y = x^na^x$. $\qquad \dfrac{dy}{dx} = a^xx^{n-1}(n+x\log a).$

54. Differentiation of sin v. Let us work out

$$\frac{d}{dv}\sin v,$$

taking, as indicated, v for the variable.

Let $$y = \sin v.$$

First step. Changing v to $v+\Delta v$, then

$$y+\Delta y=\sin(v+\Delta v).$$

Second step. $$\Delta y=\sin(v+\Delta v)-\sin v.$$

If we proceed without transforming the right-hand member, we shall encounter the same difficulty as at the beginning of Art. 51.

Applying formula 42, p. 4, by assuming

$$A=v+\Delta v, \qquad B=v,$$

and $$\therefore \tfrac{1}{2}(A+B)=v+\tfrac{1}{2}\Delta v, \qquad \tfrac{1}{2}(A-B)=\tfrac{1}{2}\Delta v,$$

we obtain

$$\Delta y=2\cos(v+\tfrac{1}{2}\Delta v)\sin\tfrac{1}{2}\Delta v.$$

Third step. $$\frac{\Delta y}{\Delta v}=\cos(v+\tfrac{1}{2}\Delta v)\cdot\frac{\sin\frac{1}{2}\Delta v}{\frac{1}{2}\Delta v},$$

in which we have written Δv beneath the second factor and multiplied numerator and denominator by $\frac{1}{2}$. If we now let Δv approach the limit zero, and apply III, p. 112, we see that the first factor approaches $\cos v$ as a limit, but that

(1) $$\operatorname*{limit}_{\Delta v=0}\frac{\sin\frac{1}{2}\Delta v}{\frac{1}{2}\Delta v}$$

cannot be found by direct substitution (p. 112). Let us therefore calculate the value of

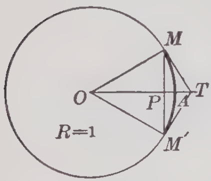
$$y=\frac{\sin x}{x},$$

into which (1) transforms if $x=\frac{1}{2}\Delta v$, for small values of x. Referring to the three-place table of Art. 4, and choosing angles less than 10°, we see that $\sin x$ and x (radians) are *equal to three decimal places.* That is, for small values of x, y equals unity very nearly. We therefore *infer* that

(2) $$\operatorname*{limit}_{x=0}\frac{\sin x}{x}=1.$$

The *proof* is the following.

In the figure, let $x = \text{arc } AM = \text{arc } AM'$, the radius OA being taken equal to unity. Then



$$\sin x = PM = PM',$$

$$\tan x = TM = TM'.$$

Now $\quad M'M < \text{arc } M'M < M'T + TM.$

$$\therefore 2 \sin x < 2x < 2 \tan x;$$

whence, dividing through by $2 \sin x$,

$$1 < \frac{x}{\sin x} < \frac{\tan x}{\sin x} \left(= \frac{1}{\cos x} \right).$$

Therefore, taking reciprocals,

$$\cos x < \frac{\sin x}{x} < 1.$$

Now let x approach zero as a limit; then, since $\cos 0 = 1$, and the value of $\frac{\sin x}{x}$ lies between 1 and $\cos x$, we must have

$$\underset{x=0}{\text{limit}} \frac{\sin x}{x} = 1.$$

The result just established enables us to complete the differentiation.

Fourth step. $\qquad \frac{dy}{dv} = \cos v.$

If v is a function of x, then by (A), Art. 50,

$$\frac{dy}{dx} = \cos v \frac{dv}{dx},$$

and we thus obtain formula

X $\qquad \boldsymbol{\frac{d}{dx} \sin v = \cos v \frac{dv}{dx}}.$

The statement of the corresponding rules will now be left to the student.

55. Differentiation of cos v.

Let $$y = \cos v.$$

By 31, p. 3, this may be written

$$y = \sin\left(\frac{\pi}{2} - v\right).$$

Differentiating by formula X,

$$\frac{dy}{dx} = \cos\left(\frac{\pi}{2} - v\right)\frac{d}{dx}\left(\frac{\pi}{2} - v\right)$$

$$= \cos\left(\frac{\pi}{2} - v\right)\left(-\frac{dv}{dx}\right)$$

$$= -\sin v\,\frac{dv}{dx}.$$

$$\left[\text{Since } \cos\left(\frac{\pi}{2} - v\right) = \sin v, \text{ by 31, p. 3.}\right]$$

XI $$\therefore \frac{d}{dx}(\cos v) = -\sin v\,\frac{dv}{dx}.$$

56. Differentiation of tan v.

Let $$y = \tan v.$$

By 27, p. 3, this may be written

$$y = \frac{\sin v}{\cos v}.$$

Differentiating by formula VII,

$$\frac{dy}{dx} = \frac{\cos v\,\dfrac{d}{dx}(\sin v) - \sin v\,\dfrac{d}{dx}(\cos v)}{\cos^2 v}$$

$$= \frac{\cos^2 v\,\dfrac{dv}{dx} + \sin^2 v\,\dfrac{dv}{dx}}{\cos^2 v}$$

$$= \frac{\dfrac{dv}{dx}}{\cos^2 v} = \sec^2 v\,\frac{dv}{dx}.$$

XII $$\therefore \frac{d}{dx}(\tan v) = \sec^2 v \frac{dv}{dx}.$$

57. Proofs of XIII–XV. By setting (using 26 and 27, p. 3)

$$\cot v = \frac{\cos v}{\sin v}, \quad \sec v = \frac{1}{\cos v}, \quad \csc v = \frac{1}{\sin v},$$

and applying VII, X, and XI, we easily prove the three formulas in question. Details are left to the student.

PROBLEMS

1. $$y = \tan \sqrt{1-x}.$$

$$\frac{dy}{dx} = \sec^2 \sqrt{1-x} \frac{d}{dx}(1-x)^{\frac{1}{2}} \qquad \text{(by XII)}$$

$$[v = \sqrt{1-x}.]$$

$$= \sec^2 \sqrt{1-x} \cdot \tfrac{1}{2}(1-x)^{-\frac{1}{2}}(-1)$$

$$= -\frac{\sec^2 \sqrt{1-x}}{2\sqrt{1-x}}.$$

2. $$y = \cos^3 x.$$

This may also be written

$$y = (\cos x)^3.$$

$$\frac{dy}{dx} = 3(\cos x)^2 \frac{d}{dx}(\cos x) \qquad \text{(by VI)}$$

$$[v = \cos x \text{ and } n = 3.]$$

$$= 3\cos^2 x(-\sin x) \qquad \text{(by XI)}$$

$$= -3\sin x \cos^2 x.$$

3. $y = \sin nx \sin^n x.$

$$\frac{dy}{dx} = \sin nx \frac{d}{dx}(\sin x)^n + \sin^n x \frac{d}{dx}(\sin nx) \qquad \text{(by V)}$$

$$[u = \sin nx \text{ and } v = \sin^n x.]$$

$$= \sin nx \cdot n(\sin x)^{n-1} \frac{d}{dx}(\sin x) + \sin^n x \cos nx \frac{d}{dx}(nx)$$

(by VI and X)

$$= n \sin nx \cdot \sin^{n-1} x \cos x + n \sin^n x \cos nx$$

$$= n \sin^{n-1} x (\sin nx \cos x + \cos nx \sin x)$$

$$= n \sin^{n-1} x \sin (n+1) x.$$

4. $y = \sec ax.$ $\quad \dfrac{dy}{dx} = a \sec ax \tan ax.$

5. $y = \tan (ax + b).$ $\quad \dfrac{dy}{dx} = a \sec^2 (ax + b).$

6. $y = \sin^2 x.$ $\quad \dfrac{dy}{dx} = \sin 2x.$

7. $y = \cos^3 x^2.$ $\quad \dfrac{dy}{dx} = -6x \cos^2 x^2 \sin x^2.$

8. $f(y) = \sin 2y \cos y.$ $\quad f'(y) = 2 \cos 2y \cos y - \sin 2y \sin y$

9. $F(x) = \cot^2 5x.$ $\quad F'(x) = -10 \cot 5x \operatorname{cosec}^2 5x.$

10. $F(\theta) = \tan \theta - \theta.$ $\quad F'(\theta) = \tan^2 \theta.$

11. $f(\phi) = \phi \sin \phi + \cos \phi.$ $\quad f'(\phi) = \phi \cos \phi.$

12. $f(t) = \sin^3 t \cos t.$ $\quad f'(t) = \sin^2 t (3 \cos^2 t - \sin^2 t).$

13. $r = a \cos 2\theta.$ $\quad \dfrac{dr}{d\theta} = -2a \sin 2\theta.$

14. $s = a \tan 2\theta.$ $\quad \dfrac{ds}{d\theta} = 2a \sec^2 2\theta.$

15. $r = a\sqrt{\cos 2\theta}.$ $\quad \dfrac{dr}{d\theta} = -\dfrac{a \sin 2\theta}{\sqrt{\cos 2\theta}}.$

16. $r = a(1 - \cos \theta).$ $\quad \dfrac{dr}{d\theta} = a \sin \theta.$

17. $r = a \sin^3 \dfrac{\theta}{3}.$ $\quad \dfrac{dr}{d\theta} = a \sin^2 \dfrac{\theta}{3} \cos \dfrac{\theta}{3}.$

18. $\dfrac{d}{dx}(\log \cos x) = -\tan x.$

19. $\dfrac{d}{dx}(\log \tan x) = \dfrac{2}{\sin 2x}.$

20. $\dfrac{d}{dx}(\log \sin^2 x) = 2 \cot x.$

21. $y = \dfrac{\tan x - 1}{\sec x}.$ $\qquad \dfrac{dy}{dx} = \sin x + \cos x.$

22. $y = \log \sqrt{\dfrac{1 + \sin x}{1 - \sin x}}.$ $\qquad \dfrac{dy}{dx} = \dfrac{1}{\cos x}.$

23. $y = \log \tan \left(\dfrac{\pi}{4} + \dfrac{x}{2}\right).$ $\qquad \dfrac{dy}{dx} = \dfrac{1}{\cos x}.$

24. $f(x) = \sin (x + a) \cos (x - a).$ $\qquad f'(x) = \cos 2x.$

25. $f(x) = \sin (\log x).$ $\qquad f'(x) = \dfrac{\cos (\log x)}{x}.$

26. $f(x) = \tan (\log x).$ $\qquad f'(x) = \dfrac{\sec^2 (\log x)}{x}.$

27. $s = \cos \dfrac{a}{t}.$ $\qquad \dfrac{ds}{dt} = \dfrac{a \sin \dfrac{a}{t}}{t^2}.$

28. $r = \sin \dfrac{1}{\theta^2}.$ $\qquad \dfrac{dr}{d\theta} = -\dfrac{2 \cos \dfrac{1}{\theta^2}}{\theta^3}.$

29. $p = \sin (\cos q).$ $\qquad \dfrac{dp}{dq} = -\sin q \cos (\cos q).$

30. $y = e^{\sin x}.$ $\qquad \dfrac{dy}{dx} = e^{\sin x} \cos x.$

31. $y = a^{\tan nx}.$ $\qquad \dfrac{dy}{dx} = n a^{\tan nx} \sec^2 nx \log a.$

32. $y = e^{\cos x} \sin x.$ $\qquad \dfrac{dy}{dx} = e^{\cos x} (\cos x - \sin^2 x).$

33. $y = e^x \log \sin x.$ $\qquad \dfrac{dy}{dx} = e^x (\cot x + \log \sin x).$

58. Inverse circular functions. The definition of arc sin x or $\sin^{-1}x$ (read "the arc sine of x" or "the inverse sine of x") is *the circular measure of the angle whose sine equals x.* Thus, if we refer to the table of Art. 4, the

$$\text{arc sin } \tfrac{1}{2} = .524, \text{ arc sin } 0 = 0, \text{ arc sin } 1 = 1.571, \text{ etc.}$$

Clearly, however, since

$$\sin 0° = 0, \sin 180° = 0, \sin 360° = 0, \text{ etc.,}$$

the value of arc sin 0 may be 0, $\pi (= 3.14)$, $2\pi (= 6.28)$ etc.; that is, arc sin 0 has an infinite number of values. To avoid this ambiguity, we make the convention:

Choose for the value of arc sin x the smallest value numerically.

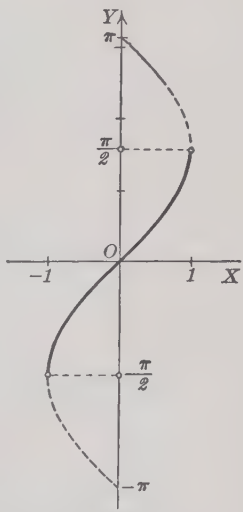
Thus $\text{arc sin } \dfrac{1}{2} = \dfrac{\pi}{3}\left(\text{not } \dfrac{5\pi}{6}\right)$, $\text{arc sin } -1 = -\dfrac{\pi}{2}\left(\text{not } \dfrac{3\pi}{2}\right)$, etc.

The graph of the equation

(1) $$y = \text{arc sin } x,$$

under the convention made, is the heavy line of the figure. For (1) is the same as

(2) $$x = \sin y.$$



Here we draw a sine curve (Art. 34) *along the y-axis.* The values of x are comprised between -1 and $+1$ inclusive, and the *smallest* corresponding values of y *numerically,* run from $-\dfrac{\pi}{2}$ to $+\dfrac{\pi}{2}$ inclusive.

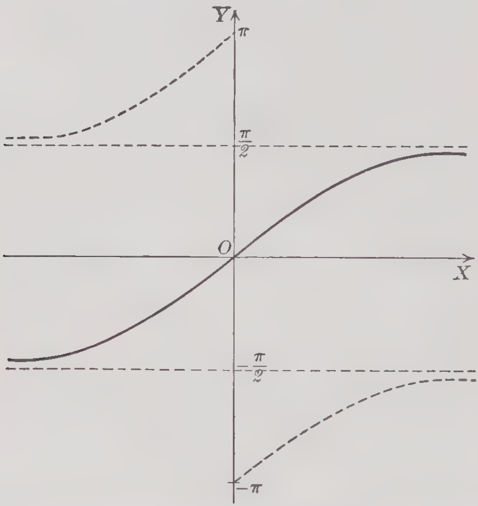
Similarly, the definition of arc tan x or $\tan^{-1}x$ is *the circular measure of the angle whose tangent equals x.* Referring to Art. 4, $\text{arc tan } 1 = \dfrac{\pi}{4}$, $\text{arc tan } 0 = 0$, $\text{arc tan } \infty = \dfrac{\pi}{2}$, etc.

To avoid ambiguity, we make the same convention as before, that is, choose the smallest value numerically.

The graph of the equation

(3) $$y = \text{arc tan } x \text{ or } x = \tan y$$

is then the heavy line of the figure. We draw a tangent curve along the y-axis, with horizontal asymptotes $y = -\frac{\pi}{2}$ and $y = \frac{\pi}{2}$.



The values of x run from $-\infty$ to $+\infty$, and the smallest corresponding values of y from $-\frac{\pi}{2}$ to $+\frac{\pi}{2}$.

59. Differentiation of $\sin^{-1} v$ and $\tan^{-1} v$. Let

(1) $$y = \text{arc sin } v.$$

Then also

(2) $$v = \sin y.$$

Differentiating with respect to y, we have

$$\frac{dv}{dy} = \cos y. \qquad \text{(by X)}$$

$$\therefore \frac{dy}{dv} = \frac{1}{\cos y}. \qquad ((5), \text{Art. } 52)$$

But (28, p. 3) $\quad \cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - v^2}. \quad$ (by (2))

(3) $$\therefore \frac{dy}{dv} = \frac{1}{\sqrt{1 - v^2}}.$$

Regarding v as a function of x, and multiplying both sides of (3) by $\frac{dv}{dx}$, we obtain (Art. 50, (A)),

(4) $$\frac{dy}{dx} = \frac{1}{\sqrt{1 - v^2}} \frac{dv}{dx},$$

which is formula XVI.

The sign of the radical in (4) is, of course, either + or −. If, however, $v = x$, (4) gives for

(5) $$y = \text{arc} \sin x, \ \frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}}.$$

By the convention made in Art. 58, the slope of the curve (5) in the figure is never negative. *Hence the sign of the radical in XVI must be taken positive.*

Let

(6) $$y = \text{arc} \tan v.$$

Then also

(7) $$v = \tan y.$$

Differentiating with respect to y,

$$\frac{dv}{dy} = \sec^2 y. \qquad (\text{by XII})$$

$$\therefore \frac{dy}{dv} = \frac{1}{\sec^2 y}. \qquad ((5), \text{Art. } 52)$$

But (28, p. 3), $\quad \sec^2 y = 1 + \tan^2 y = 1 + v^2, \quad$ (by (7))

$$\therefore \frac{dy}{dv} = \frac{1}{1 + v^2}.$$

Multiplying both members by $\frac{dv}{dx}$ gives XVII.

PROBLEMS

Differentiate the following:

1. $y = \text{arc}\sin 2x.$ $\qquad \dfrac{dy}{dx} = \dfrac{2}{\sqrt{1-4x^2}}.$

2. $y = \text{arc}\tan \frac{1}{2}x.$ $\qquad \dfrac{dy}{dx} = \dfrac{2}{4+x^2}.$

3. $y = \text{arc}\sin \dfrac{x}{a}.$ $\qquad \dfrac{dy}{dx} = \dfrac{1}{\sqrt{a^2-x^2}}.$

4. $y = \text{arc}\tan \dfrac{x}{a}.$ $\qquad \dfrac{dy}{dx} = \dfrac{a}{a^2+x^2}.$

5. $y = \text{arc}\sin \dfrac{x+1}{\sqrt{2}}.$ $\qquad \dfrac{dy}{dx} = \dfrac{1}{\sqrt{1-2x-x^2}}.$

6. $f(x) = x\sqrt{a^2-x^2} + a^2 \sin^{-1}\frac{a}{x}.$ $\qquad f'(x) = 2\sqrt{a^2-x^2}.$

7. $\theta = \sin^{-1}(3r-1).$ $\qquad \dfrac{d\theta}{dr} = \dfrac{3}{\sqrt{6r-9r^2}}.$

8. $\phi = \tan^{-1}\left(\dfrac{r+a}{1-ar}\right).$ $\qquad \dfrac{d\phi}{dr} = \dfrac{1}{1+r^2}.$

9. $\dfrac{d}{dx}(x \,\text{arc}\sin x) = \text{arc}\sin x + \dfrac{x}{\sqrt{1-x^2}}.$

10 $p = e^{\tan^{-1}q}.$ $\qquad \dfrac{dp}{dq} = \dfrac{e^{\tan^{-1}q}}{1+q^2}.$

11. $u = \tan^{-1}\dfrac{e^v - e^{-v}}{2}.$ $\qquad \dfrac{du}{dv} = \dfrac{2}{e^v - e^{-v}}.$

12. $y = \text{arc}\sin(\sin x).$ $\qquad \dfrac{dy}{dx} = 1.$

13. $y = \text{arc}\tan \dfrac{4\sin x}{3+5\cos x}.$ $\qquad \dfrac{dy}{dx} = \dfrac{4}{5+3\cos x}.$

14. $y = \text{arc}\tan\dfrac{x}{a} + \log\sqrt{\dfrac{x-a}{x+a}}.$ $\qquad \dfrac{dy}{dx} = \dfrac{2ax^2}{x^4-a^4}.$

15. $y = \log\left(\frac{1+x}{1-x}\right)^{\frac{1}{4}} - \frac{1}{2}\text{arc tan } x.$ $\qquad \frac{dy}{dx} = \frac{x^2}{1-x^4}.$

16. $y = \sqrt{1-x^2}\text{ arc sin } x - x.$ $\qquad \frac{dy}{dx} = -\frac{x \text{ arc sin } x}{\sqrt{1-x^2}}.$

17. $y = \text{arc sin } \frac{x^{2n}-1}{x^{2n}+1}.$ $\qquad \frac{dy}{dx} = \frac{2\,nx^{n-1}}{x^{2n}+1}.$

18. $\frac{d}{dx}\text{arc cos } v = \frac{-1}{\sqrt{1-v^2}}\frac{dv}{dx}.$

19. $\frac{d}{dx}\text{arc cot } v = \frac{-1}{1+v^2}\frac{dv}{dx}.$

20. $\frac{d}{dx}\sec^{-1} v = \frac{1}{v\sqrt{v^2-1}}\frac{dv}{dx}.$

60. Implicit functions. Required the slope of the curve

(1) $$x^2 - 2\,y^2 = 9.$$

If we solve for y, we shall obtain y as a function of x, namely,

(2) $$y = \sqrt{\frac{x^2-9}{2}}.$$

Instead of solving for y, however, we may in (1) regard y *implicitly* as a function of x, and differentiate *directly*. Thus, by III, we have from (1),

(3) $$\frac{d}{dx}(x^2) - \frac{d}{dx}(2\,y^2) = \frac{d}{dx}(9).$$

Remembering that y is a function of x, then

$\frac{d}{dx}(2\,y^2) = 2\frac{d}{dx}y^2 = 4\,y\frac{dy}{dx}$ (by VI). Hence (3) gives

(4) $$2\,x - 4\,y\frac{dy}{dx} = 0, \text{ or } \frac{dy}{dx} = \frac{x}{2\,y}.$$

To see that this result is the same as would be obtained by differentiating (2), using VI in (2),

$$(5) \qquad \frac{dy}{dx}=\frac{1}{2}\left(\frac{x^2-9}{2}\right)^{-\frac{1}{2}}\cdot\frac{d}{dx}\left(\frac{x^2-9}{2}\right)=\frac{x}{2\sqrt{\frac{x^2-9}{2}}}=\frac{x}{2\,y}, \text{ by (2).}$$

The general conclusion illustrated by this discussion is the following:

When the equation of a curve is given in unsolved form, either coördinate may be considered an **implicit** *function of the other. We may then differentiate directly, and solve for the desired derivative.*

Thus, to find $\frac{dy}{dx}$ from

$$x^2-3\,xy+2\,y^2=3.$$

Then $$\frac{d}{dx}(x^2)-3\frac{d}{dx}(xy)+2\frac{d}{dx}(y^2)=\frac{d}{dx}3.$$

$$\therefore 2\,x-3\left(y+x\frac{dy}{dx}\right)+4\,y\frac{dy}{dx}=0, \qquad \text{(V and VI)}$$

Solving, $$\frac{dy}{dx}=\frac{2\,x-3\,y}{3\,x-4\,y}.$$

PROBLEMS

Differentiate the following:

1. $y^2=4\,px.$ $\qquad \frac{dy}{dx}=\frac{2\,p}{y}.$

2. $x^2+y^2=r^2.$ $\qquad \frac{dy}{dx}=-\frac{x}{y}.$

3. $b^2x^2+a^2y^2=a^2b^2.$ $\qquad \frac{dy}{dx}=-\frac{b^2x}{a^2y}.$

4. $y^3-3\,y+2\,ax=0.$ $\qquad \frac{dy}{dx}=\frac{2\,a}{3(1-y^2)}.$

5. $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$. $\qquad \dfrac{dy}{dx} = -\sqrt{\dfrac{y}{x}}$.

6. $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$. $\qquad \dfrac{dy}{dx} = -\sqrt[3]{\dfrac{y}{x}}$.

7. $\left(\dfrac{x}{a}\right)^2 + \left(\dfrac{y}{b}\right)^{\frac{2}{3}} = 1$. $\qquad \dfrac{dy}{dx} = -\dfrac{3\,b^{\frac{2}{3}}xy^{\frac{1}{3}}}{a^2}$.

8. $y^2 - 2\,xy + b^2 = 0$. $\qquad \dfrac{dy}{dx} = \dfrac{y}{y - x}$.

9. $x^3 + y^3 - 3\,axy = 0$. $\qquad \dfrac{dy}{dx} = \dfrac{ay - x^2}{y^2 - ax}$.

10. $x^y = y^x$. $\qquad \dfrac{dy}{dx} = \dfrac{y^2 - xy \log y}{x^2 - xy \log x}$.

Hint. Take the logarithm of both members.

11. $\rho^2 = a^2 \cos 2\,\theta$. $\qquad \dfrac{d\rho}{d\theta} = -\dfrac{a^2 \sin 2\,\theta}{\rho}$.

12. $\rho^2 \cos\theta = a^2 \sin 3\,\theta$. $\qquad \dfrac{d\rho}{d\theta} = \dfrac{3\,a^2 \cos 3\,\theta + \rho^2 \sin\theta}{2\,\rho\cos\theta}$.

13. $\cos(uv) = cv$. $\qquad \dfrac{du}{dv} = -\dfrac{c + u\sin(uv)}{v\sin(uv)}$.

14. $\theta = \cos(\theta + \phi)$. $\qquad \dfrac{d\theta}{d\phi} = -\dfrac{\sin(\theta + \phi)}{1 + \sin(\theta + \phi)}$.

CHAPTER IX

SLOPE, TANGENT, AND NORMAL

61. If the equation of a curve is given in rectangular coördinates, it has been shown in Art. 40, p. 106, that

(1) $\quad \dfrac{dy}{dx} = \textbf{tan}\ \alpha = \textbf{slope of tangent (or curve) at } (x, y) = m.$

To find the slope at any particular point we have merely to substitute the coördinates of that point into the expression for the derivative (as on p. 109). Having thus found the numerical value of m, the equation of the tangent is given by the point-slope form, Art. 27,

(2) $$y - y_1 = m(x - x_1),$$

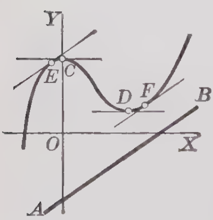
the point of contact being (x_1, y_1).

The normal is the perpendicular to the tangent drawn through the point of contact. We therefore find its equation by the formula

(3) $$y - y_1 = -\frac{1}{m}(x - x_1),$$

where m = slope of the tangent.

EXAMPLES



1. Given the curve $y = \dfrac{x^3}{3} - x^2 - 2$ (see figure).

(*a*) Find α when $x = 1$.

(*b*) Find α when $x = 3$.

(*c*) Find the points where the curve is parallel to OX.

(*d*) Find the points where $\alpha = 45°$.

(e) Find the points where the curve is parallel to the line $2x - 3y = 6$ (line AB).

Solution. Differentiating, $\frac{dy}{dx} = x^2 - 2x =$ slope at any point $= \tan \alpha$.

(a) $\tan \alpha = \left[\frac{dy}{dx}\right]_{x=1} = 1 - 2 = -1$; therefore $\alpha = 135°$. *Ans.*

(b) $\tan \alpha = \left[\frac{dy}{dx}\right]_{x=3} = 9 - 6 = 3$; therefore $\alpha = \arctan 3$. *Ans.*

(c) $\alpha = 0°$, $\tan \alpha = 0$; therefore $x^2 - 2x = 0$. Solving this equation, we find that $x = 0$ or 2, giving points C and D where curve (or tangent) is parallel to OX.

(d) $\alpha = 45°$, $\tan \alpha = 1$; therefore $x^2 - 2x = 1$. Solving, we get $x = 1 \pm \sqrt{2}$, giving two points where the slope of curve (or tangent) is unity.

(e) Slope of line $= \frac{2}{3}$; therefore $x^2 - 2x = \frac{2}{3}$. Solving, we get $x = 1 \pm \sqrt{\frac{5}{3}}$, giving points E and F where curve (or tangent) is parallel to line AB.

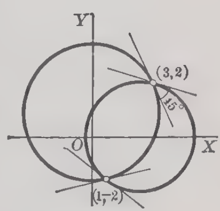
Since a curve at any point has the same direction as its tangent at that point, the angle between two curves at a common point will be the angle between their tangents at that point.

2. Find the angle of intersection of the circles

(A) $$x^2 + y^2 - 4x = 1,$$

(B) $$x^2 + y^2 - 2y = 9.$$

Solution. Solving simultaneously, we find the points of intersection to be $(3, 2)$ and $(1, -2)$.



$\frac{dy}{dx} = \frac{2 - x}{y}$ from (A). (by § 60, p. 144)

$\frac{dy}{dx} = \frac{x}{1 - y}$ from (B). (by § 60, p. 144)

$$\left[\frac{2-x}{y}\right]_{\substack{x=3\\y=2}} = -\frac{1}{2} = \text{slope of tangent to } (A) \text{ at } (3,\ 2).$$

$$\left[\frac{x}{1-y}\right]_{\substack{x=3\\y=2}} = -3 = \text{slope of tangent to } (B) \text{ at } (3,\ 2).$$

From p. 69, we know that the locus of (A) is a circle with center at $(2, 0)$ and radius $= \sqrt{5}$, and of (B) also a circle with center at $(0, 1)$ and radius $= \sqrt{10}$.

The formula for finding the angle between two lines whose slopes are m_1 and m_2 is

$$\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2}. \qquad \text{((IV), Art. 29)}$$

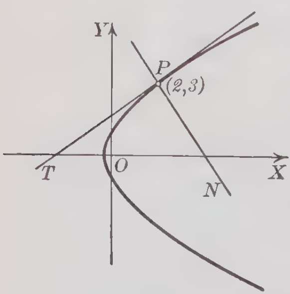
To find the smaller angle of the figure, set (p. 65) $m_1 = -\frac{1}{2}$, $m_2 = -3$.

Substituting, $\tan\theta = \dfrac{-\frac{1}{2}+3}{1+\frac{3}{2}} = 1$; therefore $\theta = 45°$. *Ans.*

This is also the angle of intersection at the point $(1, -2)$.

3. Find the equations of the tangent and normal to the parabola $y^2 = 4x + 1$ at the point where $x = 2$ and the ordinate is positive.

Solution. Substituting $x = 2$, then $y^2 = 9$, $y = \pm 3$, and the point of contact is $(2, 3)$.



Differentiating (Art. 60), we have

$$\frac{dy}{dx} = \frac{2}{y} = \text{slope at } (x,\ y).$$

$\therefore$ slope at $(2, 3) = \frac{2}{3} = m$.

Hence the required equations are

$$PT: y - 3 = \tfrac{2}{3}(x - 2), \text{ or } 2x - 3y + 5 = 0;$$

$$PN: y - 3 = -\tfrac{3}{2}(x - 2), \text{ or } 3x + 2y - 12 = 0.$$

PROBLEMS

The corresponding figure should be drawn in each of the following problems.

1. Find the slope of $y = \frac{x}{1+x^2}$ at $(0, 0)$. *Ans.* $1 = \tan \alpha$.

2. What is the direction in which the point generating the graph of $y = 3x^2 - x$ tends to move at the instant when $x = 1$?
Ans. Parallel to a line whose slope is 5.

3. Find the points where the curve $y = x^3 - 3x^2 - 9x + 5$ is parallel to the axis of X. *Ans.* $x = 3$ and $x = -1$.

4. Find the points where the curve $y(x-1)(x-2) = x - 3$ is parallel to the axis of X. *Ans.* $x = 3 \pm \sqrt{2}$.

5. At what point on $y^2 = 2x^3$ is the slope equal to 3?
Ans. $(2, 4)$.

6. At what points on the circle $x^2 + y^2 = r^2$ is the slope of tangent line equal to $-\frac{3}{4}$? *Ans.* $\pm \frac{3}{5}r, \pm \frac{4}{5}r$.

7. Where is the tangent to the parabola $y = x^2 - 7x + 3$ parallel to the line $y = 5x + 2$? *Ans.* $(6, -3)$.

8. Find the points where the tangent to the circle $x^2 + y^2 = 169$ is perpendicular to the line $5x + 12y = 60$.
Ans. $(\pm 12, \pm 5)$.

9. At what angles does the line $3y - 2x - 8 = 0$ cut the parabola $y^2 = 8x$? *Ans.* arc tan $\frac{1}{5}$, and arc tan $\frac{1}{8}$.

10. Find the angle of intersection between the parabola $y^2 = 6x$ and the circle $x^2 + y^2 = 16$. *Ans.* arc tan $\frac{5}{3}\sqrt{3}$.

11. Show that the hyperbola $x^2 - y^2 = 5$ and the ellipse $\frac{x^2}{18} + \frac{y^2}{8} = 1$ intersect at right angles.

12. At how many points can the curve $y = x^3 - 2x^2 + x - 4$ be parallel to the axis of X? What are the points?
Ans. Two; at $(1, -4)$ and $(\frac{1}{3}, -\frac{104}{27})$.

13. Find the angle at which the parabolas $y = 3x^2 - 1$ and $y = 2x^2 + 3$ intersect. *Ans.* arc tan $\frac{4}{97}$.

14. Find the equations of the tangent and normal to each of the following curves at the point indicated.

(a) $x^2+y^2=25$, $(3, 4)$. *Ans.* $3x+4y=25$, $4x-3y=0$.

(b) $y^2=2x+5$, $(2, -3)$. *Ans.* $x+3y+7=0$, $3x-y-9=0$.

(c) $y=x^3-5$, $(2, 3)$. *Ans.* $12x-y-21=0$, $x+12y-38=0$.

(d) $4y^2=x^3+8$, $(1, -\frac{3}{2})$. *Ans.* $x+4y+5=0$, $4x-y-\frac{11}{2}=0$.

(e) $x^2-4y^2=12$, $(-4, 1)$. *Ans.* $x+y+3=0$, $x-y+5=0$.

(f) $y=2\sin x$, $(0, 0)$. *Ans.* $2x-y=0$, $x+2y=0$.

(g) $y=e^x$, $(0, 1)$. *Ans.* $x-y+1=0$, $x+y-1=0$.

(h) $x^2+y^2=r^2$, (x_1, y_1). *Ans.* $x_1x+y_1y=r^2$, $y_1x-x_1y=0$.

(i) $y^2=2px$, (x_1, y_1). *Ans.* $y_1y=p(x+x_1)$,
$$y-y_1=-\frac{y_1}{p}(x-x_1).$$

(j) $y=x^3$, (x_1, y_1). *Ans.* $3x_1^2x-y-2y_1=0$,
$$x+3x_1^2y-x_1(3x_1^4+1)=0.$$

15. Plot each of the following curves, find the slope at each point plotted (p. 109), and locate all horizontal tangents.

(a) $x^2+xy+8=0$.

(b) $y^2-2xy-4=0$.

(c) $x^2y-5=0$.

(d) $y=\dfrac{x-3}{x+1}$.

(e) $y=\dfrac{1+x^2}{x}$.

(f) $y=\dfrac{x}{1+x^2}$.

(g) $y=xe^{-x}$.

(h) $y=e^{-x^2}$.

(i) $y=x^2e^{-x}$.

(j) $y=x\log x$.

(k) $y=\log x \div x$.

(l) $y=x+\sin x$.

(m) $y=\frac{1}{2}x-\cos x$.

(n) $y=x-\sin 2x$.

(o) $y=\sin x+\cos x$.

(p) $y=\sin x+\sin 2x$.

(q) $y=\cos\frac{1}{4}\pi x-\cos\frac{1}{2}\pi x$.

CHAPTER X

MAXIMA AND MINIMA

62. The graphs of the functions in Chapter VI exhibited "high points" or "low points" corresponding respectively to maximum or minimum values of the function.

A *maximum value* of a function is a value which is *greater* than all values immediately preceding or following.

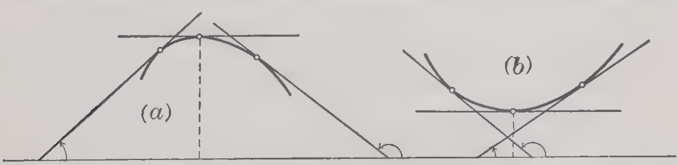
A *minimum value* of a function is a value which is *less* than all values immediately preceding or following.

Clearly, a *first condition* for a maximum or minimum value is that the slope of the graph shall be zero. That is, values of the variable for which such values of the function $f(x)$ occur, satisfy

(1) $$f'(x) = 0.$$

All such values of the variable are called **critical values.**

The statement of the problem often indicates whether a critical value corresponds to a maximum or a minimum. This was seen to be the case in the problems given, p. 96. It is easy to check up this foreseen result. For, clearly, at a high point (a) the slope is positive just before and negative just after the critical value, while reverse conditions hold for a low point (b).



The significant fact concerning maximum and minimum values of a function is, therefore, that *the derivative changes sign.*

These results are summarized as follows;

Determination of maximum and minimum values.

First step. *From the given statement find the function* (Chapter VI) *and decide, if possible, whether maximum or minimum values exist.*

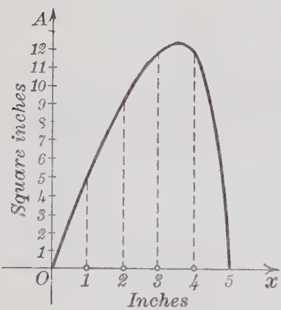
Second step. *Differentiate, set the derivative equal to zero, and solve for all real values of the variable. These are the critical values.*

Third step. *Consider any critical value not excluded by the conditions of the problem, and find the slope of the graph just before and just after the corresponding point. If the slope changes from + to −, the function is a maximum; if the reverse is true, the function is a minimum.*

The third step is to be regarded as a verification of what is foreseen in the discussion of the first step.

EXAMPLE

Given a circle of diameter 5 in. Examine the areas of inscribed rectangles for maxima and minima (see p. 92).



Solution. *First step.* If x represents the base of any rectangle of area A, then (1), p. 92,

$$A = x\sqrt{25 - x^2}. \tag{1}$$

Discussion shows that a maximum exists.

Second step. Differentiating,

$$\frac{dA}{dx} = \sqrt{25 - x^2} + x\frac{d}{dx}(25 - x^2)^{\frac{1}{2}},$$

$$= \sqrt{25 - x^2} - \frac{x^2}{\sqrt{25 - x^2}};$$

$$\therefore \frac{dA}{dx} = \frac{25 - 2x^2}{\sqrt{25 - x^2}}, \tag{2}$$

Setting this equal to zero, we obtain $25 - 2x^2 = 0$, and hence the critical value is $x = \frac{5}{\sqrt{2}} = 3.55$, the negative value being excluded.

Third step. For a value just before the critical value, use $x = 3$. Then from (2), $\left[\frac{dA}{dx}\right]_{x=3} = +$ number. For a value just greater, use $x = 4$. Then $\left[\frac{dA}{dx}\right]_{x=4} = -$ number. Thus the result foreseen in the first step is verified. That is, the maximum rectangle has the area $A = \frac{5}{\sqrt{2}}\sqrt{25 - \frac{25}{2}} = \frac{25}{2} = 12\frac{1}{2}$ sq. in. It is easy to see that the rectangle is now a square. For when $x = \frac{5}{\sqrt{2}}$, the altitude $= \sqrt{25 - x^2} = \frac{5}{\sqrt{2}}$ (p. 92) $= x$.

Hence this example proves the result:—

Of all rectangles which can be inscribed in a given circle, the square has the greatest area.

PROBLEMS

In the following problems the student will work out the functional relation, and examine this for maxima and minima.

1. Rectangles are inscribed in a circle of radius r. Examine the perimeter P of the rectangles as a function of the breadth x. *Ans.* Max. for $x = r\sqrt{2}$.

2. Right triangles are constructed on a line of given length h as hypotenuse. Examine (*a*) the area A, and (*b*) the perimeter P as a function of the length x of one leg.

Ans. (*a*) Max. for $x = \frac{1}{2}h\sqrt{2}$. (*b*) Max. for $x = \frac{1}{2}h\sqrt{2}$.

3. Right cylinders are inscribed in a sphere of radius r. Examine as functions of the altitude x of the cylinder, (*a*) volume V of the cylinder, (*b*) curved surface S.

Ans. (*a*) Max. for $x = \frac{2}{3}r\sqrt{3}$. (*b*) Max. for $x = r\sqrt{2}$.

4. Right cones are inscribed in a sphere of radius r. Examine as functions of the altitude x of the cone, (a) volume V of the cone, (b) curved surface S.

Ans. (a) Max. if $x = \frac{4}{3}r$. (b) Max. if $x = \frac{4}{3}r$.

5. Right cylinders are inscribed in a given right cone. If the height of the cone is h, and the radius of the base r, examine (a) the volume V of the cylinder, (b) the curved surface S, (c) the entire surface T, as functions of the altitude x of the cylinder. *Ans.* (a) Max. if $x = \frac{1}{3}h$. (b) Max. if $x = \frac{1}{2}h$.

6. Right cones are circumscribed about a sphere of radius r. Examine as a function of the altitude x of the cone, the volume V of the cone. *Ans.* Min. if $x = 4r$.

7. Right cones are constructed with a given slant height L. Examine as functions of the altitude x of the cone, (a) the volume V of the cone, (b) the curved surface S, (c) the entire surface T. *Ans.* (a) $V = $ Max. if $x = \frac{1}{3}L\sqrt{3}$. (b) Neither.

8. A conical tent is to be constructed of given volume V. Examine the amount A of canvas required as a function of the radius x of the base. *Ans.* Min. if $x = \frac{1}{2}\sqrt{2}$ altitude.

9. A cylindrical tin can is to be constructed of given volume V. Examine the amount A of tin required as a function of the diameter x of the can. *Ans.* Min. if $2x =$ altitude.

10. An open box is to be made from a sheet of pasteboard 12 in. square, by cutting equal squares from the four corners and bending up the sides. Examine the volume V as a function of the side x of the square cut out.

Ans. Max. if $x = 2$.

11. The strength of a rectangular beam is proportional to the product of the cross section by the square of the depth. Examine the strength S as a function of the depth x for beams which are cut from a log 12 in. in diameter.

Ans. Max. if $x = 6\sqrt{3}$.

12. A rectangular stockade is to be built to contain a certain area A. A stone wall already constructed is available for one of the sides. Examine the length L of the wall to be built as a function of the length x of the side of the rectangle parallel to the wall. *Ans.* Min. if $x =$ twice other side.

13. A tower is 100 ft. high. Examine the angle α subtended by the tower at a point on the ground as a function of the distance x from the foot of the tower. *Ans.* Neither.

14. A tower 50 ft. high is surmounted by a statue 10 ft. high. If an observer's eyes are in a horizontal plane with the base, examine the angle α subtended by the statue as a function of the observer's distance x from the tower.

Ans. Max. if $x = 10\sqrt{30}$.

15. A line is drawn through a fixed point (a, b). Examine, as a function of the intercept on XX' $(= x)$ of the line, the area A of the triangle formed with the coördinate axes.

Ans. Min. when $x = 2a$.

16. A ship is 41 mi. due north of a second ship. The first sails south at the rate of 8 mi. an hour, the second east at the rate of 10 mi. an hour. Examine their distance d apart as a function of the time t which has elapsed since they were in the position given. *Ans.* Min. if $t = 2$.

17. Examine the distance e from the point $(4, 0)$ to the points (x, y) on the parabola $y^2 = 4x$. *Ans.* Min. if $x = 2$.

18. A gutter is to be constructed whose cross section is a broken line made up of three pieces each 4 in. long, the middle piece being horizontal, and the two sides being equally inclined. Examine the area A of a cross section of the gutter as a function of the width x of the gutter across the top.

Ans. Max. for $x = 8$.

19. A Norman window consists of a rectangle surmounted by a semicircle. Given the perimeter P, examine the area A as a function of the width x. *Ans.* Max. when $x =$ total height.

20. A person in a boat 9 mi. from the nearest point of the beach wishes to reach a place 15 mi. from that point along the shore. He can row at the rate of 4 mi. an hour and walk at the rate of 5 mi. an hour. The time it takes him to reach his destination depends on the place at which he lands. Examine the time. *Ans.* Min. if he lands 3 mi. from the camp.

21. The illumination of a plane surface by a luminous point varies directly as the cosine of the angle of incidence, and inversely as the square of the distance from the surface. Examine the illumination I of a point on the floor 10 ft. from the wall, as a function of the height x of a gas burner on the wall. *Ans.* Max. if $x = 5\sqrt{2}$.

22. The sides of a quadrilateral are given in order and length. When is the area a maximum?

23. Examine the functions of problems 22–30, p. 99, for maxima and minima.

63. Derivatives of higher orders. Since the derivative of a function of a variable x with respect to x is also in general a function of x, we may differentiate the derivative itself, that is, carry out the operation,

$$\frac{d}{dx}\left(\frac{d}{dx}f(x)\right).$$

This double operation is indicated by the more compact notation,

$$\frac{d^2}{dx^2}f(x),$$

and this new function is called the *second derivative*. In the same way,

$$\frac{d}{dx}\frac{d^2}{dx^2}f(x) \equiv \frac{d^3}{dx^3}f(x)$$

is the *third derivative*, and in general,

$$\frac{d^n}{dx^n}f(x)$$

is the nth derivative of $f(x)$, that is, the result of differentiating $f(x)$ n times. The following notation is also used:

$$\frac{d}{dx}f(x)=f'(x),\ \frac{d^2}{dx^2}f(x)=f''(x),\ \cdots,\ \frac{d^n}{dx^n}f(x)=f^{(n)}(x).$$

The operation of finding the successive derivatives is called *successive differentiation.*

For example,

given $$f(x)=3x^4-4x^2+6x-1,$$

then $$f'(x)=12x^3-8x+6;$$

$$f''(x)=36x^2-8, \text{ etc.}$$

If the independent variable is y, then the second differentiation gives the value of

$$\frac{d}{dx}\left(\frac{dy}{dx}\right)\equiv\frac{d^2y}{dx^2},$$

the abbreviation indicated being that commonly employed. The symbol $\frac{d^2y}{dx^2}$ is read, "the second derivative of y with respect to x." Similarly,

$$\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)\equiv\frac{d^3y}{dx^3},$$

is the third derivative, etc.

Thus, given $$y=x^3-4x^2.$$

Then $$\frac{dy}{dx}=3x^2-8x;$$

$$\frac{d^2y}{dx^2}=6x-8;$$

$$\frac{d^3y}{dx^3}=6;$$

$$\frac{d^4y}{dx^4}=0; \text{ etc}$$

PROBLEMS

Differentiate the following.

1. $y = 4x^3 - 6x^2 + 4x + 7.$ $\qquad \dfrac{d^2y}{dx^2} = 12(2x-1).$

2. $f(x) = \dfrac{x^3}{1-x}.$ $\qquad f^{\text{iv}}(x) = \dfrac{\lfloor 4}{(1-x)^5}.$

3. $f(y) = y^6.$ $\qquad f^{\text{vi}}(y) = \lfloor 6.$

4. $y = x^3 \log x.$ $\qquad \dfrac{d^4y}{dx^4} = \dfrac{6}{x}.$

5. $y = \dfrac{c}{x^n}.$ $\qquad \dfrac{d^2y}{dx^2} = \dfrac{n(n+1)c}{x^{n+2}}.$

6. $y = (x-3)e^{2x} + 4xe^x + x.$ $\qquad \dfrac{d^2y}{dx^2} = 4e^x[(x-2)e^x + x + 2]$

7. $y = \dfrac{a}{2}(e^{\frac{x}{a}} + e^{-\frac{x}{a}}).$ $\qquad \dfrac{d^2y}{dx^2} = \dfrac{1}{2a}(e^{\frac{x}{a}} + e^{-\frac{x}{a}}) = \dfrac{y}{a^2}.$

8. $f(x) = ax^2 + bx + c.$ $\qquad f'''(x) = 0.$

9. $f(x) = \log(x+1).$ $\qquad f^{\text{v}}(x) = -\dfrac{6}{(x+1)^4}.$

10. $f(x) = \log(e^x + e^{-x}).$ $\qquad f'''(x) = -\dfrac{8(e^x - e^{-x})}{(e^x + e^{-x})^3}.$

11. $r = \sin a\theta.$ $\qquad \dfrac{d^4r}{d\theta^4} = a^4 \sin a\theta = a^4 r.$

12. $r = \tan\phi.$ $\qquad \dfrac{d^3r}{d\phi^3} = 6\sec^4\phi - 4\sec^2\phi.$

13. $r = \log\sin\phi.$ $\qquad \dfrac{d^3r}{d\phi^3} = 2\cot\phi\,\text{cosec}^2\,\phi.$

14. $f(t) = e^{-t}\cos t.$ $\qquad f^{\text{iv}}(t) = -4e^{-t}\cos t = -4f(t).$

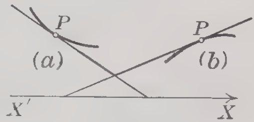
15. $f(\theta) = \sqrt{\sec 2\theta}.$ $\qquad f''(\theta) = 3[f(\theta)]^5 - f(\theta).$

64. Geometrical significance of the second derivative. It has been observed that the value of the first derivative of a

function determines the slope of the graph. Moreover, we have seen, in Art. 38, that

(i) if the derivative is *positive*, the function *increases* as the variable increases;

(ii) if the derivative is *negative*, the function *decreases* as the variable increases.

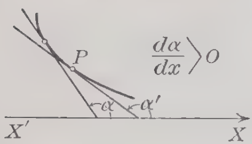


If we consider the appearance of a curve in the neighborhood of one of its points P, then clearly two cases are distinguishable:

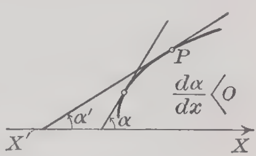
(a) The curve is *above* the tangent at P, or is **concave upward**;

(b) The curve is *below* the tangent at P, or is **concave downward.**

How are these cases distinguishable analytically?



Consider (a). As we approach P from the left, that is, with increasing values of x, we observe from the figure that the inclination α of the tangent *increases* ($\alpha' > \alpha$). That is, in (a), the inclination α increases as x increases. Hence by (i), if $\frac{d\alpha}{dx}$ is positive at P, the curve is certainly concave upward.



Consider (b). Here, α *decreases* as x increases ($\alpha' < \alpha$). Hence if $\frac{d\alpha}{dx} < 0$, the curve is concave downward.

Summing up, if

(A) $\frac{d\alpha}{dx} > 0$, *curve is concave upward;* $\frac{d\alpha}{dx} < 0$, *concave downward.*

Now, if $f'(x)$ is the derivative of the function $f(x)$, then

(2) $$f'(x) = \tan\alpha, \text{ or } \alpha = \text{arc} \tan f'(x).$$

Differentiating with respect to x, using XVII,

(3) $$\frac{d\alpha}{dx}=\frac{\frac{d}{dx}f'(x)}{1+f'^2(x)}=\frac{f''(x)}{1+f'^2(x)},$$

where $f''(x)=$ second derivative of $f(x)$. Hence (3) becomes

(4) $$\frac{d\alpha}{dx}=\frac{f''(x)}{1+f'^2(x)}.$$

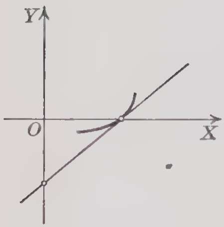
Since the denominator $1+f'^2(x)$ is always positive, $\frac{d\alpha}{dx}$ has the same sign as $f''(x)$, and conversely. Hence the result:

second derivative >0, curve concave upward;

second derivative <0, curve concave downward.

As an example, consider this question:

EXAMPLE



Is the curve $y=x\log x$ concave upward or downward at $x=1$?

Solution. Differentiating twice,

$$\frac{dy}{dx}=\log x+1,$$

$$\frac{d^2y}{dx^2}=\frac{1}{x}.$$

Hence when $x=1$, $\frac{d^2y}{dx^2}=1=$ a positive number. Hence the curve is concave upward at $x=1$.

From the first derivative, $\left.\frac{dy}{dx}\right]_{x=1}=\log 1+1=1.$ Also from the equation of the curve, when $x=1$, $y=\log 1=0$. Arranged in a table, we have the results set down.

x	y	$\frac{dy}{dx}$	$\frac{d^2y}{dx^2}$
1	0	1	+

The *appearance* of the curve, in the *neighborhood* of $x=1$, is now determined, and we may construct a *small* arc of the curve at that point, as in the figure.

PROBLEMS

1. Determine the *direction of curvature* (concave upward or downward) of the following curves at the points indicated. Draw a figure in each case.

(*a*) $y = \sin x,\qquad x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{2}.$

(*b*) $y = 3x - x^3,\qquad x = 1, 2, -3.$

(*c*) $y = e^{-x},\qquad x = 0, -1, 2.$

(*d*) $y = xe^{-x},\qquad x = 0, 3.$

(*e*) $y = e^{-x}\cos x,\qquad x = 0.$

(*f*) $y = \log(1 + x),\qquad x = 0, 2.$

(*g*) $y = \frac{1 + x^2}{x},\qquad x = 1, -1, 2.$

2. Show that each of the following curves is always concave upward or downward.

(*a*) $y = a\log x.$ (*b*) $y = 4x - x^2.$ (*c*) $y = ae^{-x}.$

65. Second test for maxima and minima. At a high point, the graph is concave downward; at a low point, concave upward. The directions on p. 152 may therefore be stated in another form, the first and second steps, however, remaining unchanged.

Determination of maxima and minima — second method.

First step. *Same as on p.* 152.

Second step. *Same as on p.* 152.

Third step. *Consider any critical value not excluded by the conditions of the problem, and substitute this in the second derivative of the function. If the result is negative, the function has a maximum value; if positive, a minimum value; if zero, the method of p.* 152 *must be resorted to.*

Clearly, this second method should be used only if the second derivative is easily obtained. For example, the method of p. 152 is preferable for the problem solved on p. 152. On the other hand, if we turn to the problem on p. 94, equation (2),

$$M = x^2 + \frac{432}{x},$$

the differentiation is simple, namely,

$$\frac{dM}{dx} = 2x - \frac{432}{x^2}, \qquad \text{(see p. 105)}$$

$$\frac{d^2M}{dx^2} = 2 + \frac{864}{x^3}.$$

We may conveniently test the critical value

$$x = 6 \left(\frac{dM}{dx} = 0 \text{ for } x = 6\right)$$

by substitution in the second derivative. This gives $\left.\frac{d^2M}{dx^2}\right]_{x=6}$ = a positive number. Hence M is a minimum when $x = 6$.

In each of the following problems the function is easily differentiated and the second method should be adopted.

PROBLEMS

1. A circular sector has a given perimeter. Show that the area is a maximum when the angle of the sector is 2 radians. (Area of a sector equals $\frac{1}{2}$ arc times radius.)

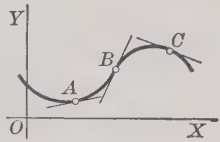
2. A Gothic window consists of a rectangle surmounted by an equilateral triangle. The perimeter is given. What must be the proportions in order to admit as much light as possible?

3. Apply the second test for maxima and minima to problems 3 (a), 4 (a), 5, 6, 7 (a), 9, 10, 12, 14, 15, and 19 of Art. 62.

66. Points of inflection. The problems on p. 110 illustrate the aid afforded by the use of the first derivative in obtaining

an accurate plot. The second derivative is also of service in this connection. For if we know the sign of the second derivative at any point on the curve, we then know if the curve at that point is concave upward or concave downward (p. 159), and we therefore know if the curve at that point lies *above* or *below* the tangent.

Suppose we have a continuous curve (as in this figure) such that in passing along it from A to C, the curve *changes* from concave upwards to concave downwards. Then a point B must exist such that



to the left of B, the curve is concave upwards;
to the right of B, the curve is concave downwards.

The point B is called a *point of inflection.*

The second derivative $\left(f''(x) \text{ or } \frac{d^2y}{dx^2}\right)$ is *positive* at each point of the arc AB, and negative at each point of the arc BC. Hence at B the second derivative is zero.

(B) At points of inflection, $f''(x)$ or $\frac{d^2y}{dx^2}$ equals zero.

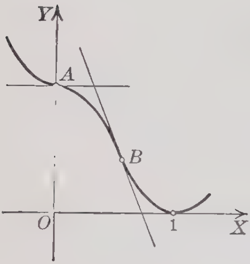
Solving the equation resulting from (B) gives the abscissas of the points of inflection. To determine the direction of curving in the vicinity of a point of inflection, test $f''(x)$ for values of x, first a trifle less and then a trifle greater than the abscissa at that point.

If $f''(x)$ changes sign, we have a point of inflection, and the signs obtained determine if the curve is concave upwards or concave downwards in the neighborhood of each point.

The student should observe that near a point where the curve is concave upwards (as at A) the curve lies above the tangent, and at a point where the curve is concave downwards (as at C) the curve lies below the tangent. At a point of inflection (as at B) the tangent evidently crosses the curve.

PROBLEMS

Examine the following curves and graphs for horizontal tangents, points of inflection, and direction of curving.



1. $y = 3x^4 - 4x^3 + 1.$

Solution. $f(x) = 3x^4 - 4x^3 + 1.$

Hence

$$f'(x) = 12x^3 - 12x^2 = 12x^2(x-1).$$

$\therefore f'(x) = 0$ when $x = 0$, or $x = 1$.

Differentiating again,

$$f''(x) = 36x^2 - 24x.$$

Using (B),

$$36x^2 - 24x = 0.$$

$$\therefore x = \tfrac{2}{3} \text{ and } x = 0.$$

Factoring, $\qquad f''(x) = 36x(x - \tfrac{2}{3}).$

When $x < 0, f''(x) = +$; and when $x > 0, f''(x) = -$.

$\therefore$ the curve is concave upwards to the left and concave downwards to the right of $x = 0$ (A in figure).

When $x < \frac{2}{3}, f''(x) = -$; and when $x > \frac{2}{3}, f''(x) = +$.

$\therefore$ the curve is concave downwards to the left and concave upwards to the right of $x = \frac{2}{3}$ (B in figure).

The curve is evidently concave upwards everywhere to the left of A, concave downwards between A $(0, 1)$ and B $(\frac{2}{3}, \frac{11}{27})$, and concave upwards everywhere to the right of B.

In work of this kind it is well to tabulate the results which afford a check on the plot. We follow the plot from *left to right*, and choose for the initial value of x *a value to the left of the least root* of $f'(x) = 0$ and $f''(x) = 0$. The table should include all critical values of x $(f'(x) = 0)$ and also those for which $f''(x) = 0$, that is, values determining the points of inflection. Furthermore, intermediate values of x must be included, as in the following table, which the student should carefully study.

x	y	$\frac{dy}{dx}$	$\frac{d^2y}{dx^2}$	Remarks
-1	8	24	$+$	Concave upward
0	1	0	0	Point of inflection Horizontal tangent
$\frac{1}{2}$	$\frac{11}{16}$	$-\frac{3}{2}$	$-$	Concave downward
$\frac{2}{3}$	$\frac{11}{27}$	$-\frac{16}{9}$	0	Point of inflection Slope of tangent $= -\frac{16}{9}$
1	0	0	$+$	Minimum value of y
2	17	48	$+$	Concave upward

2. $y = x^2$. *Ans.* Concave upwards everywhere.

3. $y = 5 - 2x - x^2$. *Ans.* Concave downwards everywhere.

4. $y = x^3$. *Ans.* Concave downwards to the left and concave upwards to the right of $(0, 0)$.

5. $y = x^3 - 3x^2 - 9x + 9$. *Ans.* Concave downwards to the left and concave upwards to the right of $(1, -2)$.

6. $y = a + (x - b)^3$. *Ans.* Concave downwards to the left and concave upwards to the right of (a, b).

7. $a^2y = \frac{x^3}{3} - ax^2 + 2a^3$. *Ans.* Concave downwards to the left and concave upwards to the right of $\left(a, \frac{4a}{3}\right)$.

8. $x^3 - 3bx^2 + a^2y = 0$. *Ans.* Point of inflection is $\left(b, \frac{2b^3}{a^2}\right)$.

9. $y = x^4$. *Ans.* Concave upwards everywhere.

10. $3x^3 - 9x^2 - 27x + 30$. *Ans.* $x = -1$, gives max. $= 45$;
$x = 3$, gives min. $= -51$.

11. $2x^3 - 21x^2 + 36x - 20$. *Ans.* $x = 1$, gives max. $= -3$;
$x = 6$, gives min. $= -128$.

12. $\dfrac{x^3}{3} - 2x^2 + 3x + 1$. *Ans.* $x = 1$, gives max. $= \frac{7}{3}$;
$x = 3$, gives min. $= 1$.

13. $2x^3 - 15x^2 + 36x + 10$. *Ans.* $x = 2$, gives max. $= 38$;
$x = 3$, gives min. $= 37$.

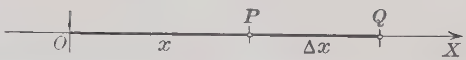
14. $x^3 - 9x^2 + 15x - 3$. *Ans.* $x = 1$, gives max. $= 4$;
$x = 5$, gives min. $= -28$.

15. $x^3 - 3x^2 + 6x + 10$. *Ans.* No max. or min.

16. $x^5 - 5x^4 + 5x^3 + 1$. *Ans.* $x = 1$, gives max. $= 2$;
$x = 3$, gives min. $= -26$;
$x = 0$, gives neither.

CHAPTER XI

RATES



67. Velocity and acceleration. Consider a moving point on the x-axis. Its distance from the origin ($= x$) is a function of the time. That is, symbolically,

$$(1) \qquad x = f(t).$$

Suppose that P is the position when the time is t seconds, and suppose, further, that the elapsed time from P to Q is Δt seconds. Also let $PQ = \Delta x$. Then the quotient

$$(2) \qquad \frac{\Delta x}{\Delta t} = \textit{average velocity of the motion from } P \textit{ to } Q.$$

The *velocity at* P is **the limit** of the value of $\frac{\Delta x}{\Delta t}$ when Q is taken nearer and nearer to Q; that is

$$(3) \qquad \frac{dx}{dt} = \underset{\Delta t = 0}{\text{limit}} \frac{\Delta x}{\Delta t} = \textbf{velocity at } P.$$

Considered without the idea of motion, equation (1) asserts that the variable x changes as t changes. Then we say

$$(4) \qquad \frac{dx}{dt} = \textit{time rate of change of } x.$$

Similarly, if y and z are functions of the time, then $\frac{dy}{dt}$ and $\frac{dz}{dt}$ are the time rates of change of y and z respectively.

The point P may move from the point O with constant

speed v_0. Then the distance moved is the product of the speed (or velocity) and the time. That is, equation (1) now is

$$x = v_0 t. \tag{5}$$

If, however, the velocity changes, the time rate of change of velocity is called the *acceleration*, and equation (4) asserts that

$$\textbf{acceleration} = \frac{dv}{dt}. \tag{6}$$

From (3) we may also write,

$$\text{acceleration} = \frac{dv}{dt} = \frac{d^2x}{dt^2}. \tag{7}$$

Similarly, any derivative may be interpreted as a rate of change. For example, $\frac{dy}{dx}$ = rate of change of y with respect to x, if y is a function of x.

The case frequently arises when in the equation

$$f(x, y) = 0$$

both x and y are functions of the time. *If the time rate of change in* x $\left(= \frac{dx}{dt}\right)$ *is known, then* $\frac{dy}{dt}$ *can be found.*

PROBLEMS

1. A particle slides along the curve, $y^2 - 4x = 0$, so that it moves in the XX' direction at the constant rate of 3 ft. a second. How fast is it moving in the YY' direction (*a*) at any point (x, y); (*b*) at the point (4, 4)?

Solution. Given $\frac{dx}{dt} = 3$, required $\frac{dy}{dt}$.

As y and x in the equation $y^2 - 4x = 0$ are functions of t, we differentiate the equation with respect to t. This gives (p. 144),

$$2y\frac{dy}{dt} - 4\frac{dx}{dt} = 0,$$

Solving for $\frac{dy}{dt}$,

(a) $\frac{dy}{dt}=\frac{2}{y}\frac{dx}{dt}=\frac{6}{y}$ ft. a second; (b) $\frac{dy}{dt}=\frac{6}{4}=\frac{3}{2}$ ft. a second.

2. Find the rate of change of the volume and of the surface of a sphere with respect to the radius.

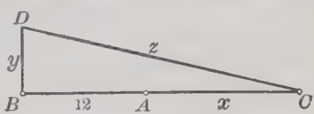
Solution. Let y be the volume and x the radius. Required $\frac{dy}{dx}$.

Now $$y=\frac{4}{3}\pi x^3, \therefore \frac{dy}{dx}=4\pi x^2,$$

which varies as the square of the radius. The rate of change of the surface z with respect to the radius x is found from the relation $z=4\pi x^2$ to be $\frac{dz}{dx}=8\pi x$, and it varies as the first power of the radius.

3. An express train and a balloon start at points 12 mi. apart. The former runs 50 mi. per hour, and the balloon rises vertically at the rate of 8 mi. per hour. How fast are they separating at the end of 10 min.?

Solution. Let the figure represent the state of affairs at the expiration of *any number* ($=t$) of minutes. That is,



BD is the path of the balloon,
AC is the path of the train,

and $y=$ number of miles traveled by balloon,
$x=$ number of miles traveled by train,
$z=$ distance apart.

Evidently, x, y, and z are all functions of the time.

Now get the relation between these variables.

Evidently,

(1) $$z^2=y^2+(x+12)^2.$$

Differentiate with respect to t. This gives

$$2z\frac{dz}{dt}=2y\frac{dy}{dt}+2(x+12)\frac{dx}{dt}.$$

(2) $$\therefore\ \frac{dz}{dt}=\left(y\frac{dy}{dt}+(x+12)\frac{dx}{dt}\right)\div z.$$

This result gives the velocity of separation at *any time.* The problem calls for this velocity after 10 min. Hence

$$x=8\tfrac{1}{3}\text{ miles},\ y=1\tfrac{1}{3}\text{ miles},\ \frac{dx}{dt}=50,\ \frac{dy}{dt}=8.$$

We find from (1) the value of $z=20.4$ mi. Substituting in (2) we find $\frac{dz}{dt}=50.3$ mi. *Ans.*

This problem shows that in many cases the following rule should be followed.

1. *Draw a diagram to represent the state of affairs at any instant (after t seconds).*
2. *In this diagram mark the variable elements with x, y, z, etc.*
3. *Find the equation connecting x, y, z, etc.*
4. *Differentiate this result with respect to t.*
5. *Substitute the given data in this result.*

4. The radius of a soap bubble is increasing at the rate of 2 in. a second. How fast is the volume increasing, (*a*) at any time, (*b*) when the radius is 3 in.?

Ans. (*a*) $8\pi r^2$ cu. in. a second, (*b*) 72π cu. in. a second.

5. Solve the problem similar to the preceding, where volume is replaced by surface.

6. At what point of Ex. 1 are the ordinate and the abscissa increasing at the same rate? *Ans.* (1, 2).

7. Where in the first quadrant does the arc increase twice as fast as the sine? *Ans.* 60°.

8. A man 6 ft. tall walks away from a lamp post 10 ft. high at the constant rate of 4 mi. an hour. How fast does the shadow of his head move? (Use similar triangles to determine the relation between the man's distance from the lamp post and the distance of the shadow.)

Ans. 10 mi. an hour.

9. Find the rate of change of the area of a square, when the side x is increasing at the rate of k in. a second.

Ans. $2\,kx$ sq. in. a second.

10. A ladder 25 ft. long rests against a house. A man takes hold of the lower end and moves it away at the rate of 2 ft. a second. How fast is the top of the ladder descending when the bottom is 7 ft. from the house? *Ans.* 7 in. a second.

11. Two particles start together from the origin along $y^2 - 9\,x = 0$ and $x^2 + y^2 - 34\,x = 0$ respectively. The former has a speed in the YY' direction of 3 ft. a second, and the latter of 4 ft. a second. Which goes through the point (25, 15) with the greater speed in the XX' direction?

Ans. One on the parabola.

12. Find how fast the man's shadow in problem 8 is lengthening. *Ans.* 6 mi. an hour.

13. A circular plate expands by heat so that the radius increases uniformly at the rate of .01 in. a minute. At what rate is the surface increasing when the radius is 2 in.?

Ans. .126 sq. in. a minute.

14. Water runs into a barrel at the rate of 2 cu. ft. a minute, but leaks at the bottom at the rate of 1 cu. ft. a minute. Assuming the barrel to be a right cylinder of radius 1 ft. and of height 5 ft., how long will it be before water runs over the top? *Ans.* 15 min. 42 sec.

15. A boy starts flying a kite. If it moves horizontally at the rate of 2 ft. a second, and rises at the rate of 5 ft. a second, how fast is the string being paid out at the end of 3 min.?

Ans. 5.38 ft. a second.

16. At what point of the curve $xy + 25 = 0$ will a particle move in the x and y direction at the same rate? *Ans.* $(-5, 5)$.

17. A man standing on a dock is drawing in the painter of a boat at the rate of 2 ft. a second. His hands are 6 ft. above the bow of the boat. How fast is the boat moving when it is 8 ft. from the dock? *Ans.* $\frac{5}{2}$ ft. a second.

18. The volume and the radius of a cylindrical boiler are expanding at the rate of 1 cu. ft. and .001 ft. per minute respectively. How fast is the length of the boiler changing when the boiler contains 60 cu. ft. and has a radius of 2 ft.?

Ans. .074 ft. a minute.

19. An equilateral triangular sheet of rubber is stretched so that it always keeps its shape, but expands at the rate of 3 sq. in. a minute. How fast is its side increasing when 4 in. long?

Ans. .866 in. a minute.

20. The rays of the sun make an angle of 30° with the horizon. A ball is thrown vertically upward to a height of 64 ft. How fast is its shadow traveling along the ground just before the ball hits the ground? (Use $s = \frac{1}{2}gt^2$ and $v = gt$.)

Ans. 110.8 ft. a second.

21. A man is walking over a bridge at the rate of 4 mi. an hour, and a motor boat passes beneath him at the rate of 8 mi. an hour. If the bridge is 20 ft. above the boat, how fast are they separating 3 min. later? *Ans.* 8.95 mi. an hour.

22. A ship is anchored in 18 ft. of water. The cable passes over a sheave on the bow 6 ft. above the surface of the water. If the cable is taken in at the rate of 1 ft. a second, how fast is the ship moving when there are 30 ft. of cable out?

Ans. $\frac{5}{3}$ ft. a second.

23. A boy rows out for a swim against a tide running in at the rate of $\frac{1}{2}$ mi. an hour horizontally. He dives off and swims parallel to the coast at the rate of 2 mi. an hour. How fast are he and his boat separating at the end of 15 min.?

Ans. 2.06 mi. an hour.

24. Four men standing 5 ft. from a house are hoisting a piano to the third-story window by means of a block and tackle. If the window is 50 ft. up, and the men pull in the rope at the rate of 10 ft. a minute and back away from the building at the rate of 5 ft. a minute, how fast is the piano rising at the end of the first minute? *Ans.* 10.98 ft. a minute.

25. Find at what points on $y = x^2 - 4$ the rate of increase of y with respect to x is equal to the rate of increase of x with respect to y. *Ans.* $(\pm \frac{1}{2}, -\frac{15}{4})$.

26. Assuming the volume of wood in a tree is proportional to the cube of its diameter and that the latter increases uniformly year by year, show that the rate of growth when the diameter is 3 ft. is 36 times as great as when it is 6 in.

27. A rectangular sheet of metal is subjected to a pressure which expands it only lengthwise and at a rate of 2 in. a minute. Find how fast the area of the sheet is enlarging, if it is 7 in. wide. *Ans.* 14 sq. in. a minute.

28. Water flows from a faucet into a hemispherical basin of diameter 14 in. at the rate of 2 cu. in. a second. How fast is the water rising (a) when the water is halfway to the top, (b) just as it runs over? (The volume of a spherical segment is $\frac{1}{2}\pi r^2 h + \frac{1}{6}\pi h^3$.)

Ans. (a) .021 in. a second; (b) .013 in. a second.

29. Sand is being poured on the ground from the orifice of an elevated pipe, and forms a pile which has always the shape of a right circular cone whose height is equal to the radius of the base. If sand is falling at the rate of 6 cu. ft. per second, how fast is the height of the pile increasing when the height is 5 ft.? *Ans.* .076 ft. per second.

30. An aeroplane is 528 ft. directly above an automobile, and starts east at the rate of 20 mi. an hour at the same time that the automobile starts east at the rate of 40 mi. an hour. How fast are they separating in 6 min.?

Ans. 19.92 mi. an hour.

31. A ship is 41 mi. due north of a second ship. The first sails south at the rate of 8 mi. an hour; the second, east, at the rate of 10 mi. an hour. How rapidly are they approaching each other? How long will they continue to approach?

Ans. For 2 hr.

32. A railroad train is running along a curve in the form of $y^2 = 500\,x$, the axis of the parabola being east and west. If the train is going due east at the rate of 30 mi. an hour, how fast is the shadow moving along the wall of a station which runs north and south, when the train is 500 ft. east of the wall, provided the sun is just rising in the east?

Ans. 15 mi. an hour.

33. One side of a rectangle inscribed in a circle of radius 5 cm. expands at the rate of 2 cm. a minute. Find (*a*) how fast the area of the rectangle is changing when the above side is 8 cm.; (*b*) how long the above side is when the area is not changing. (Get the area as a function of the side.)

Ans. (*a*) $-9\frac{1}{3}$ cm.^{2} a minute; (*b*) 7.07.

34. Work out similar problems for cylinders and cones inscribed in a fixed sphere, where the radii of their bases vary.

35. A revolving light sending out a bundle of parallel rays is at a distance of $\frac{1}{2}$ a mile from the shore and makes 1 revolution a minute. Find how fast the light is traveling along the beach when at a distance of 1 mi. from the nearest point of the beach. *Ans.* 15.7 mi. per minute.

36. A kite is 150 ft. high, and 200 ft. of string are out. If the kite starts drifting horizontally and away from the flyer, at the rate of 4 mi. an hour, how fast is string being paid out?

Ans. 2.64 mi. an hour.

37. A solution is poured into a conical filter of base radius 6 cm. and height 24 cm. at the rate of 2 cu. cm. a second, which filters out at the rate of 1 cu. cm. a second. How fast is the level of the solution rising when (*a*) one third of the way up, (*b*) at the top? *Ans.* (*a*) .079 cm. a second,

(*b*) .009 cm. a second.

CHAPTER XII

DIFFERENTIALS

68. We have already emphasized that the symbol

$$\frac{dy}{dx}$$

was to be considered not as an ordinary fraction with dy as numerator and dx as denominator, but as a single symbol denoting the limit of the quotient

$$\frac{\Delta y}{\Delta x}$$

as Δx approaches the limit zero.

Problems do occur, however, where it is convenient to give meanings to dx and dy separately. How this may be done is explained in what follows.

Definition. If $f'(x)$ is the derivative of $f(x)$, and Δx is an arbitrarily chosen increment of x, then the *differential of* $f(x)$, denoted by the symbol $df(x)$, is defined by the equation

$$(A) \qquad df(x) = f'(x)\,\Delta x.$$

If now $f(x) = x$, then $f'(x) = 1$, and (A) reduces to

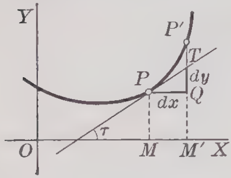
$$dx = \Delta x.$$

Hence, when x is the independent variable, the *differential of* $x (= dx)$ is identical with Δx. If $y = f(x)$, (A) may be written in the form

$$(\boldsymbol{B}) \qquad \boldsymbol{dy = f'(x)\,dx.}*$$

* On account of the position which the derivative $f'(x)$ here occupies, it is sometimes called the *differential coefficient.*

The differential of a function equals its derivative multiplied by the differential of the independent variable.



Let us illustrate what this means geometrically.

Let $f'(x)$ be the derivative of $y=f(x)$ at P.

Take $dx = PQ$, then

$$dy = f'(x)\,dx = \tan\tau \cdot PQ$$

$$= \frac{QT}{PQ} \cdot PQ = QT.$$

Therefore dy, or $df(x)$, is the increment ($= QT$) of the ordinate of the tangent corresponding to dx.

This gives the following interpretation of the derivative as a fraction:

If an arbitrarily chosen increment of the independent variable x for a point $P(x, y)$ on the curve $y=f(x)$ be denoted by dx, then in the derivative

$$\frac{dy}{dx} = f'(x) = \tan\tau$$

dy denotes the corresponding increment of the ordinate of the tangent to the curve at the point P.

The student should observe that, while the differential and the increment of the independent variable are always equal ($dx = \Delta x$), the same is not true of the dependent variable. In the figure, $\Delta y =$ increment of $y = QP'$, but $dy =$ differential of $y = QT$.

69. Formulas for finding differentials. Since the differential of a function is its derivative multiplied by the differential of the independent variable, it follows at once that the formulas for finding differentials may be obtained from those for finding derivatives (Art. 42) by multiplying by dx.

This gives

I $\qquad d(c) = 0.$

II $\qquad d(x) = dx.$

III $$d(u+v-w)=du+dv-dw.$$

IV $$d(cv)=cdv.$$

V $$d(uv)=u\,dv+v\,du.$$

VI $$d(v^n)=nv^{n-1}\,dv.$$

VI a $$d(x^n)=nx^{n-1}\,dx.$$

VII $$d\left(\frac{u}{v}\right)=\frac{v\,du-u\,dv}{v^2}.$$

VII a $$d\left(\frac{u}{c}\right)=\frac{du}{c}.$$

VIII $$d(\log_a v)=\log_a e\frac{dv}{v}.$$

IX $$d(a^v)=a^v\log a\,dv.$$

IX a $$d(e^v)=e^v\,dv.$$

X $$d(\sin v)=\cos v\,dv.$$

XI $$d(\cos v)=-\sin v\,dv.$$

XII $$d(\tan v)=\sec^2 v\,dv, \text{ etc.}$$

XVI $$d(\arcsin v)=\frac{dv}{\sqrt{1-v^2}}, \text{ etc.}$$

The term *differentiation* also includes the operation of finding differentials.

In finding differentials, the easiest way is to find the derivative as usual, and then multiply the result by dx.

EXAMPLES

1. Find the differential of

$$y=\frac{x+3}{x^2+3}.$$

Solution.

$$dy=d\left(\frac{x+3}{x^2+3}\right)=\frac{(x^2+3)\,d(x+3)-(x+3)\,d(x^2+3)}{(x^2+3)^2}$$

$$=\frac{(x^2+3)dx-(x+3)2x\,dx}{(x^2+3)^2}=\frac{(3-6x-x^2)\,dx}{(x^2+3)^2}. \quad \textit{Ans.}$$

2. Find dy from

$$b^2x^2 - a^2y^2 = a^2b^2.$$

Solution.

$$2\,b^2xdx - 2\,a^2ydy = 0.$$

$$\therefore\ dy = \frac{b^2x}{a^2y}\,dx. \quad Ans.$$

3. Find $d\rho$ from

$$\rho^2 = a^2 \cos 2\,\theta.$$

Solution. $$2\,\rho d\rho = -\,a^2 \sin 2\,\theta \cdot 2\,d\theta.$$

$$\therefore\ d\rho = -\,\frac{a^2 \sin 2\,\theta}{\rho}\,d\theta.$$

4. Find $d[\text{arc}\sin(3\,t - 4\,t^3)]$.

Solution.

$$d[\text{arc}\sin(3\,t - 4\,t^3)] = \frac{d(3\,t - 4\,t^3)}{\sqrt{1-(3\,t-4\,t^3)^2}} = \frac{3\,dt}{\sqrt{1-t^2}}. \quad Ans.$$

PROBLEMS

Differentiate the following, using differentials.

1. $y = ax^3 - bx^2 + cx + d.$ $\qquad dy = (3\,ax^2 - 2\,bx + c)dx.$

2. $y = 2\,x^{\frac{5}{2}} - 3\,x^{\frac{2}{3}} + 6x^{-1} + 5.$ $\qquad dy = (5\,x^{\frac{3}{2}} - 2\,x^{-\frac{1}{3}} - 6\,x^{-2})dx.$

3. $y = (a^2 - x^2)^5.$ $\qquad dy = -\,10\,x(a^2 - x^2)^4 dx.$

4. $y = \sqrt{1 + x^2}.$ $\qquad dy = \dfrac{x}{\sqrt{1+x^2}}\,dx.$

5. $y = \dfrac{x^{2n}}{(1+x^2)^n}.$ $\qquad dy = \dfrac{2\,nx^{2n-1}}{(1+x^2)^{n+1}}\,dx.$

6. $y = \log\sqrt{1 - x^3}.$ $\qquad dy = \dfrac{3\,x^2dx}{2(x^3-1)}.$

7. $y = (e^x + e^{-x})^2.$ $\qquad dy = 2(e^{2x} - e^{-2x})dx.$

8. $y = e^x \log x.$ $\qquad dy = e^x\left(\log x + \dfrac{1}{x}\right)dx.$

9. $s = t - \dfrac{e^t - e^{-t}}{e^t + e^{-t}}.$ $\qquad ds = \left(\dfrac{e^t - e^{-t}}{e^t + e^{-t}}\right)^2 dt.$

10. $\rho = \tan\phi + \sec\phi$. $\qquad d\rho = \dfrac{1+\sin\phi}{\cos^2\phi}d\phi$.

11. $r = \frac{1}{3}\tan^3\theta + \tan\theta$. $\qquad dr = \sec^4\theta d\theta$.

12. $f(x) = (\log x)^3$. $\qquad f'(x)dx = \dfrac{3(\log x)^2 dx}{x}$.

13. $\phi(t) = \dfrac{t^3}{(1-t^2)^{\frac{3}{2}}}$. $\qquad \phi'(t)dt = \dfrac{3t^2 dt}{(1-t^2)^{\frac{5}{2}}}$.

70. Infinitesimals. An infinitesimal is a variable whose value decreases numerically and approaches zero as a limit.

In Art. 68 it has been shown that the differential and increment of the *independent* variable are identical. Equation (B) of that section defines the differential of the dependent variable. Clearly, if dx is an infinitesimal, then also dy is an infinitesimal. In the Integral Calculus, we have to do with expressions of the form $\phi(x)dx$, called "differential expressions." Then, if dx is an infinitesimal, so also is the product $\phi(x)dx$.

CHAPTER XIII

INTEGRATION. RULES FOR INTEGRATING STANDARD ELEMENTARY FORMS

71. Integration. The student is already familiar with the mutually inverse operations of addition and subtraction, multiplication and division, involution and evolution. From the Differential Calculus we have learned how to calculate the derivative $f'(x)$ of a given function $f(x)$, an operation indicated by

$$\frac{d}{dx}f(x)=f'(x),$$

or, if we are using differentials, by

$$df(x)=f'(x)\,dx.$$

The problems of the Integral Calculus depend on the *inverse operation;* namely,

To find a function $f(x)$ whose derivative

$$(A)\qquad f'(x)=\phi(x)$$

is given.

Or, since it is customary to use differentials in the Integral Calculus, we may write

$$(B)\qquad df(x)=f'(x)\,dx=\phi(x)\,dx,$$

and state the problem as follows:

Having given the differential of a function, to find the function itself.

The function $f(x)$ thus found is called an *integral* of the given differential expression, the process of finding it is called *inte-*

gration, and the operation is indicated by writing the *integral sign* * $\int$ in front of the given differential expression. Thus,

$$(C) \qquad \int f'(x)\,dx = f(x),$$

read, *an integral of* $f'(x)\,dx$ *equals* $f(x)$. The differential dx indicates that x is *the variable of integration*. For example,

(*a*) If $f(x) = x^3$, then $\quad f'(x)\,dx = 3\,x^2dx,$

and
$$\int 3\,x^2dx = x^3.$$

(*b*) If $f(x) = \sin x$, then $\quad f'(x)\,dx = \cos x dx,$

and
$$\int \cos x dx = \sin x.$$

(*c*) If $f(x) = \text{arc} \tan x$, then $\quad f'(x) = \dfrac{dx}{1+x^2},$

and
$$\int \frac{dx}{1+x^2} = \text{arc} \tan x.$$

Let us now emphasize what is apparent from the preceding explanations, namely, that

Differentiation and integration are inverse operations.

Differentiating (C) gives

$$(D) \qquad d\int f'(x)\,dx = f'(x)\,dx.$$

Substituting the value of $f'(x)\,dx\,[=df(x)]$ from (B) in (C), we get

$$(E) \qquad \int df(x) = f(x).$$

Therefore, considered as symbols of operation, $\dfrac{d}{dx}$ and $\int \cdots dx$

* Historically this sign is a distorted S, the initial letter of the word *sum*. Instead of defining integration as the inverse of differentiation we may define it as a process of summation, a very important notion which we consider in Chapter XVI.

are *inverse to each other;* or, if we are using differentials, d and $\int$ are inverse to each other.

When d is followed by $\int$ they annul each other, as in (D), but when $\int$ is followed by d, as in (E), that will not in general be the case unless we ignore the *constant of integration.* The reason for this will appear at once from the definition of the constant of integration given in the next section.

72. Constant of integration. Indefinite integral. From the preceding section it follows that

$$\text{since } d(x^3) = 3\,x^2dx, \text{ we have } \int 3\,x^2dx = x^3;$$

$$\text{since } d(x^3+2) = 3\,x^2dx, \text{ we have } \int 3\,x^2dx = x^3+2;$$

$$\text{since } d(x^3-7) = 3\,x^2dx, \text{ we have } \int 3\,x^2dx = x^3-7.$$

In fact, since $\qquad d(x^3+C) = 3\,x^2dx,$

where C is any arbitrary constant, we have

$$\int 3\,x^2dx = x^3 + C.$$

A constant C arising in this way is called a *constant of integration.** Since we can give C as many values as we please, it follows that if a given differential expression has one integral, it has infinitely many differing only by constants. Hence

$$\int f'(x)\,dx = f(x) + C;$$

and since C is unknown and *indefinite*, the expression

$$f(x) + C$$

is called the *indefinite integral of $f'(x)\,dx$.*

* Constant here means that it is independent of the *variable of integration.*

It is evident that if $\phi(x)$ is a function the derivative of which is $f(x)$, then $\phi(x)+C$, where C is any constant whatever, is likewise a function the derivative of which is $f(x)$. Hence the

Theorem. *If two functions differ by a constant, they have the same derivative.*

73. Rules for integrating standard elementary forms. The Differential Calculus furnished us with a *General Rule* for Differentiation (p. 104). The Integral Calculus gives us no corresponding general rule that can be readily applied in practice for performing the inverse operation of integration.

Each case requires special treatment, and we arrive at the integral of a given differential expression through our previous knowledge of the known results of differentiation.

Integration then is essentially a tentative process, and to expedite the work, tables of known integrals are formed called *standard forms.* To effect any integration we compare the given differential expression with these forms, and if it is found to be identical with one of them, the integral is known. If it is not identical with one of them, we strive to reduce it to one of the standard forms by various methods, many of which employ artifices which can be suggested by practice only.

From any result of differentiation may always be derived a formula for integration.

The following two rules are useful in reducing differential expressions to standard forms.

(*a*) *The integral of any algebraic sum of differential expressions equals the same algebraic sum of the integrals of these expressions taken separately.*

Proof. Differentiating the expression

$$\int du + \int dv - \int dw,$$

u, v, w being functions of a single variable, we get

$$du + dv - dw, \qquad \text{(III, p. 113)}$$

[1] $$\therefore \int (du + dv - dw) = \int du + \int dv - \int dw.$$

(*b*) *A constant factor may be written either before or after the integral sign.*

Proof. Differentiating the expression

$$a \int dv$$

gives $$adv.$$ (IV, p. 113)

[2] $$\therefore \int adv = a \int dv.$$

On account of their importance we shall write the above two rules as formulas at the head of the following list of

STANDARD ELEMENTARY FORMS

[1] $$\int (du + dv - dw) = \int du + \int dv - \int dw.$$

[2] $$\int adv = a \int dv.$$

[3] $$\int dx = x + C.$$

[4] $$\int v^n dv = \frac{v^{n+1}}{n+1} + C. \qquad n \neq -1$$

[5] $$\int \frac{dv}{v} = \log v + C$$
$$= \log v + \log c = \log cv.$$
[Placing $C = \log c$.]

[6] $$\int a^v dv = \frac{a^v}{\log a} + C.$$

[7] $$\int e^v dv = e^v + C.$$

[8] $$\int \sin v \, dv = - \cos v + C.$$

[9] $$\int \cos v \, dv = \sin v + C.$$

[10] $$\int \sec^2 v \, dv = \tan v + C.$$

[11] $$\int \operatorname{cosec}^2 v\, dv = -\cot v + C.$$

[12] $$\int \sec v \tan v\, dv = \sec v + C.$$

[13] $$\int \csc v \cot v\, dv = -\csc v + C.$$

[14] $$\int \tan v\, dv = \log \sec v + C.$$

[15] $$\int \cot v\, dv = \log \sin v + C.$$

[16] $$\int \frac{dv}{v^2 + a^2} = \frac{1}{a} \operatorname{arc\,tan} \frac{v}{a} + C.$$

[17] $$\int \frac{dv}{\sqrt{a^2 - v^2}} = \operatorname{arc\,sin} \frac{v}{a} + C.$$

Proof of [3]. Since

$$d(x + C) = dx, \qquad \text{(II, p. 113)}$$

then $$\int dx = x + C.$$

Proof of [4]. Since

$$d\left(\frac{v^{n+1}}{n+1}\right) = v^n dv, \qquad \text{(VII, p. 113)}$$

then $$\int v^n dv = \frac{v^{n+1}}{n+1} + C.$$

This holds true for all values of n except $n = -1$. For, when $n = -1$, [4] gives

$$\int v^{-1} dv = \frac{v^{-1+1}}{-1+1} + C = \frac{1}{0} + C = \infty + C,$$

which has no meaning.

The case when $n = -1$ comes under [5].

Proof of [5]. Since

$$d(\log v + C) = \frac{dv}{v}, \qquad \text{(VIII } a\text{, p. 113)}$$

we get $$\int \frac{dv}{v} = \log v + C.$$

The results we get from [5] may be put in more compact form if we denote the constant of integration by $\log c$. Thus,

$$\int \frac{dv}{v} = \log v + \log c = \log cv.$$

Formula [5] states that *if the expression under the integral sign is a fraction whose numerator is the differential of the denominator, then the integral is the natural logarithm of the denominator.*

PROBLEMS

For formulas [1]–[5].

Verify the following integrations.

1. $\int x^6 dx = \frac{x^7}{7} + C$, by [4], where $v = x$ and $n = 6$.

2. $\int ax^3 dx = a \int x^3 dx$ (by [2])

$$= \frac{ax^4}{4} + C. \qquad \text{(by [4])}$$

3. $\int (2x^3 - 5x^2 - 3x + 4)\, dx$

$$= \int 2x^3 dx - \int 5x^2 dx - \int 3x dx + \int 4 dx \quad \text{(by [1])}$$

$$= 2\int x^3 dx - 5\int x^2 dx - 3\int x dx + 4\int dx \quad \text{(by [2])}$$

$$= \frac{x^4}{2} - \frac{5x^3}{3} - \frac{3x^2}{2} + 4x + C.$$

Note. Although each separate integration requires an arbitrary constant, we write down only a single constant denoting their algebraic sum.

4. $\int \left(\frac{2a}{\sqrt{x}} - \frac{b}{x^2} + 3c\sqrt[3]{x^2} \right) dx$

$$= 2a\int x^{-\frac{1}{2}} dx - b\int x^{-2} dx + 3c \int x^{\frac{2}{3}} dx \quad \text{(by [1] and [2])}$$

$$= 2a \cdot \frac{x^{\frac{1}{2}}}{\frac{1}{2}} - b \cdot \frac{x^{-1}}{-1} + \frac{3c \cdot x^{\frac{5}{3}}}{\frac{5}{3}} + C \quad \text{(by [4])}$$

$$= 4a\sqrt{x} + \frac{b}{x} + \tfrac{9}{5} cx^{\frac{5}{3}} + C.$$

5. $\int 2\,ax^{b-1}dx = \frac{2\,ax^b}{b} + C.$ 6. $\int 3\,mz^6 dz = \frac{3\,mz^7}{7} + C.$

7. $\int \left(bs^3 + \frac{1}{s^{\frac{3}{2}}}\right)ds = \frac{bs^4}{4} - \frac{2}{\sqrt{s}} + C.$

8. $\int \sqrt{2\,px}\,dx = \frac{2}{3}\,x\sqrt{2\,px} + C.$

9. $\int (a^{\frac{2}{3}} - x^{\frac{2}{3}})^3\,dx = a^2x - \frac{9}{7}\,a^{\frac{2}{3}}x^{\frac{7}{3}} + \frac{9}{5}\,a^{\frac{4}{3}}x^{\frac{5}{3}} - \frac{x^3}{3} + C.$

Hint. First expand.

10. $\int (a^2 - y^2)^3 \sqrt{y}\,dy = 2\,y^{\frac{3}{2}}\left(\frac{a^6}{3} - \frac{3\,a^4y^2}{7} + \frac{3\,a^2y^4}{11} - \frac{y^6}{15}\right) + C.$

11. $\int (\sqrt{a} - \sqrt{t})^3 dt = a^{\frac{3}{2}}t - 2\,at^{\frac{3}{2}} + \frac{3\,a^{\frac{1}{2}}t^2}{2} - \frac{2\,t^{\frac{5}{2}}}{5} + C.$

12. $\int \frac{dx}{(nx)^{\frac{n-1}{n}}} = (nx)^{\frac{1}{n}} + C.$

13. $\int (a^2 + b^2x^2)^{\frac{1}{2}}\,xdx = \frac{(a^2 + b^2x^2)^{\frac{3}{2}}}{3\,b^2} + C.$

Solution. This may be brought into the form [4]. For let $v = a^2 + b^2x^2$ and $n = \frac{1}{2}$; then $dv = 2\,b^2xdx$. If we now insert the constant factor $2\,b^2$ before xdx, and its reciprocal $\frac{1}{2\,b^2}$ before the integral sign (so as not to change the value of the expression), the expression may be integrated, using [4], namely,

$$\int v^n dv = \frac{v^{n+1}}{n+1} + C.$$

Thus,
$$\int (a^2 + b^2x^2)^{\frac{1}{2}}\,xdx = \frac{1}{2\,b^2}\int (a^2 + b^2x^2)^{\frac{1}{2}}\,2\,b^2xdx$$
$$= \frac{1}{2\,b^2}\int (a^2 + b^2x^2)^{\frac{1}{2}}\,d(a^2 + b^2x^2)$$
$$= \frac{1}{2\,b^2} \cdot \frac{(a^2 + b^2x^2)^{\frac{3}{2}}}{\frac{3}{2}} + C = \frac{(a^2 + b^2x^2)^{\frac{3}{2}}}{3\,b^2} + C.$$

Note. The student is warned against transferring any function of the variable from one side of the integral sign to the other, since that would change the value of the integral.

14. $\int\sqrt{a^2 - x^2}\,xdx = -\frac{1}{3}(a^2 - x^2)^{\frac{3}{2}} + C.$

15. $\int(3\,ax^2 + 4\,bx^3)^{\frac{4}{3}}(2\,ax + 4\,bx^2)dx = \frac{1}{7}(3\,ax^2 + 4\,bx^3)^{\frac{7}{3}} + C.$

Hint. Use [4], making $v = 3\,ax^2 + 4\,bx^3$ and $n = \frac{4}{3}$.

16. $\int b(6\,ax^2 + 8\,bx^3)^{\frac{5}{3}}(2\,ax + 4\,bx^2)dx = \frac{b}{16}(6\,ax^2 + 8\,bx^3)^{\frac{8}{3}} + C.$

17. $\int\frac{x^2dx}{(a^2 + x^3)^{\frac{1}{2}}} = \frac{2}{3}(a^2 + x^3)^{\frac{1}{2}} + C.$

Hint. Write this $\int(a^2 + x^3)^{-\frac{1}{2}}x^2dx$ and apply [4].

18. $\int\frac{dx}{\sqrt{1 - x}} = -2\sqrt{1 - x} + C.$

19. $\int 2\,\pi y\left(\frac{y^2}{p^2} + 1\right)^{\frac{1}{2}}dy = \frac{2\,\pi}{3\,p}(y^2 + p^2)^{\frac{3}{2}} + C.$

20. $\int\frac{2\,asds}{(b^2 - c^2s^2)^2} = \frac{a}{c^2(b^2 - c^2s^2)} + C.$

21. $\int\frac{3\,axdx}{b^2 + e^2x^2} = \frac{3\,a}{2\,e^2}\log(b^2 + e^2x^2) + C.$

Solution. $\int\frac{3\,axdx}{b^2 + e^2x^2} = 3\,a\int\frac{xdx}{b^2 + e^2x^2} + C.$

This resembles [5]. For let $v = b^2 + e^2x^2$; then $dv = 2\,e^2xdx$. If we introduce the factor $2\,e^2$ after the integral sign, and $\frac{1}{2\,e^2}$ before it, we have not changed the value of expression, and the numerator is now seen to be the differential of the denominator. Therefore

$$3\,a\int\frac{xdx}{b^2 + e^2x^2} = \frac{3\,a}{2\,e^2}\int\frac{2\,e^2xdx}{b^2 + e^2x^2} = \frac{3\,a}{2\,e^2}\int\frac{d\,(b^2 + e^2x^2)}{b^2 + e^2x^2}$$
$$= \frac{3\,a}{2\,e^2}\log(b^2 + e^2x^2) + C. \qquad \text{(by [5])}$$

22. $\int\frac{xdx}{x^2 - 1} = \frac{1}{2}\log(x^2 - 1) + \log C = \log c\sqrt{x^2 - 1}.$

23. $\displaystyle\int\frac{(x^2-a^2)\,dx}{x^3-3\,a^2x}=\log c(x^3-3\,a^2x)^{\frac{1}{3}}.$

24. $\displaystyle\int\frac{5\,x^2dx}{10\,x^3+15}=\log c(10\,x^3+15)^{\frac{1}{6}}.$

25. $\displaystyle\int\frac{5\,xdx}{8\,a-6\,bx^2}=\log\frac{c}{(8\,a-6\,bx^2)^{\frac{5}{12}}}.$

26. $\displaystyle\int\frac{x^3dx}{x+1}=x-\frac{x^2}{2}+\frac{x^3}{3}-\log(x+1)+C.$

Hint. First divide the numerator by the denominator.

27. $\displaystyle\int\frac{2\,x-1}{2\,x+3}\,dx=x-\log(2\,x+3)^2+C.$

28. $\displaystyle\int\frac{x^{n-1}-1}{x^n-nx}\,dx=\frac{1}{n}\log(x^n-nx)+C.$

29. $\displaystyle\int\frac{(y^2-2)^3dy}{y^5}=\frac{2}{y^4}-\frac{6}{y^2}+\frac{y^2}{2}-\log y^6+C.$

30. $\displaystyle\int\frac{t^{n-1}dt}{a+bt^n}=\frac{1}{nb}\log(a+bt^n)+C.$

Proofs of [6] *and* [7]. These follow at once from the corresponding formulas for differentiation, IX and IX a, p. 113.

PROBLEMS

For formulas [6] and [7].

Verify the following integrations.

1. $\displaystyle\int ba^{2x}dx=\frac{ba^{2x}}{2\log a}+C.$

Solution. $\displaystyle\int ba^{2x}dx=b\int a^{2x}dx.$ (by [2])

This resembles [6]. Let $v=2\,x$; then $dv=2\,dx$. If we then insert the factor 2 before dx and the factor $\frac{1}{2}$ before the integral sign, we have,

$$b\int a^{2x}dx=\frac{b}{2}\int a^{2x}2\,dx=\frac{b}{2}\int a^{2x}d(2\,x)=\frac{b}{2}\cdot\frac{a^{2x}}{\log a}+C.$$

(by [6])

2. $\int 3\, e^x dx = 3\, e^x + C.$

3. $\int (e^{5x} + a^{5x})\, dx = \frac{1}{5}\left(e^{5x} + \frac{a^{5x}}{\log a}\right) + C.$

4. $\int e^{\frac{x}{n}} dx = n e^{\frac{x}{n}} + C.$

5. $\int e^{x^2+4x+3}(x+2)\, dx = \frac{1}{2}\, e^{x^2+4x+3} + C.$

6. $\int (a^{nx} - b^{mx})\, dx = \frac{a^{nx}}{n \log a} - \frac{b^{mx}}{m \log b} + C.$

7. $\int a^x e^x dx = \frac{a^x e^x}{1 + \log a} + C.$

8. $\int (e^{\frac{x}{a}} + e^{-\frac{x}{a}})\, dx = a\,(e^{\frac{x}{a}} - e^{-\frac{x}{a}}) + C.$

9. $\int (e^y + e^{-y})^2 dy = \frac{1}{2}\,(e^{2y} - e^{-2y}) + 2\, y + C.$

10. $\int (3\, e^{2t} - 1)^{\frac{1}{3}} e^{2t} dt = \frac{1}{8}\,(3\, e^{2t} - 1)^{\frac{4}{3}} + C.$

Proofs of [8]–[13]. These follow at once from the corresponding formulas for differentiation, X, etc., p. 113.

Proof of [14].

$$\int \tan v dv = \int \frac{\sin v dv}{\cos v} = -\int \frac{-\sin v dv}{\cos v}$$

$$= -\int \frac{d\,(\cos v)}{\cos v}$$

$$= -\log \cos v + C \qquad \text{(by [5])}$$

$$= \log \sec v + C.$$

$$\left[\text{Since } -\log \cos v = -\log \frac{1}{\sec v} = -\log 1 + \log \sec v = \log \sec v.\right]$$

Proof of [15]. $\int \cot v dv = \int \frac{\cos v dv}{\sin v} = \int \frac{d\,(\sin v)}{\sin v}$

$$= \log \sin v + C. \qquad \text{(by [5])}$$

PROBLEMS

For formulas [8]–[17].

Verify the following integrations.

1. $\int \sin 2\,ax dx = -\frac{\cos 2\,ax}{2\,a} + C.$

Solution. This resembles [8]. For let $v = 2\,ax$, then $dv = 2\,adx$. If we now insert the factor $2\,a$ before dx, and the factor $\frac{1}{2\,a}$ before the integral sign, we get

$$\int \sin 2\,ax dx = \frac{1}{2\,a}\int \sin 2\,ax \cdot 2\,adx$$

$$= \frac{1}{2\,a}\int \sin 2\,ax \cdot d(2\,ax) = \frac{1}{2\,a} \cdot -\cos 2\,ax + C \qquad \text{(by [8])}$$

$$= -\frac{\cos 2\,ax}{2\,a} + C.$$

2. $\int \cos mx dx = \frac{1}{m}\sin mx + C.$

3. $\int 5\sec^2 bx dx = \frac{5}{b}\tan bx + C.$

4. $\int \left(\cos\frac{\theta}{3} - \sin 3\,\theta\right)d\theta = 3\sin\frac{\theta}{3} + \frac{1}{3}\cos 3\,\theta + C.$

5. $\int 7\sec 3\,\alpha\tan 3\,\alpha d\alpha = \frac{7}{3}\sec 3\,\alpha + C.$

6. $\int k\cos(a + by)\,dy = \frac{k}{b}\sin(a + by) + C.$

7. $\int \operatorname{cosec}^2 x^3 \cdot x^2 dx = -\frac{1}{3}\cot x^3 + C.$

8. $\int 4\csc ax\cot ax dx = -\frac{4}{a}\csc ax + C.$

9. $\int \frac{\sin x dx}{a + b\cos x} = \log\frac{c}{(a + b\cos x)^{\frac{1}{b}}}.$

10. $\int e^{\cos x} \sin x dx = - e^{\cos x} + C.$

11. $\int \frac{dx}{\cos^2(a - bx)} = -\frac{1}{b}\tan(a - bx) + C.$

12. $\int \cos(\log x)\frac{dx}{x} = \sin\log x + C.$

13. $\int \frac{dx}{\sin^2\frac{x}{n}} = -n\cot\frac{x}{n} + C.$

14. $\int \frac{(1 + \cos x)\,dx}{x + \sin x} = \log(x + \sin x)C.$

15. $\int \frac{\sin\phi\,d\phi}{\cos^2\phi} = \sec\phi + C.$

16. $\int \sin^2 x dx = \frac{1}{2}x - \frac{1}{4}\sin 2x + C.$

[Substitute $\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x$.]

17. $\int \cos^2 x dx = \frac{1}{2}x + \frac{1}{4}\sin 2x + C.$

[Substitute $\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x$.]

18. $\int \tan^2 x dx = \tan x - x + C.$

[Substitute $\tan^2 x = \sec^2 x - 1$.]

19. $\int \cot^2 x dx = -\cot x - x + C.$

[Substitute $\cot^2 x = \operatorname{cosec}^2 x - 1$.]

Proof of [16]. Since

$$d\left(\operatorname{arc}\tan\frac{v}{a} + C\right) = \frac{d\left(\frac{v}{a}\right)}{1 + \left(\frac{v}{a}\right)^2} = \frac{adv}{v^2 + a^2}, \quad \text{(by XVII, p. 114)}$$

we get

$$\int \frac{dv}{v^2 + a^2} = \frac{1}{a}\operatorname{arc}\tan\frac{v}{a} + C.$$

Proof of [17]. Since

$$d\left(\arc \sin \frac{v}{a} + C\right) = \frac{d\left(\frac{v}{a}\right)}{\sqrt{1-\left(\frac{v}{a}\right)^2}} = \frac{dv}{\sqrt{a^2 - v^2}}, \quad \text{(by XVI, p. 114)}$$

we get $\displaystyle\int \frac{dv}{\sqrt{a^2 - v^2}} = \text{arc} \sin \frac{v}{a} + C.$

PROBLEMS

For formulas (16) and (17).

Verify the following integrations.

1. $\displaystyle\int \frac{dx}{4x^2+9} = \frac{1}{6} \text{arc} \tan \frac{2x}{3} + C.$

Solution. This resembles [16]. For, let $v^2 = 4x^2$ and $a^2 = 9$; then $v = 2x$, $dv = 2\,dx$, and $a = 3$. Hence if we multiply the numerator by 2, we get

$$\int \frac{dx}{4x^2+9} = \frac{1}{2}\int \frac{2\,dx}{(2x)^2 + (3)^2} = \frac{1}{2}\int \frac{d(2x)}{(2x)^2 + (3)^2}$$

$$= \frac{1}{6} \text{arc} \tan \frac{2x}{3} + C. \qquad \text{(by (16))}$$

2. $\displaystyle\int \frac{dx}{\sqrt{16 - 9x^2}} = \frac{1}{3} \text{arc} \sin \frac{3x}{4} + C.$

3. $\displaystyle\int \frac{5x\,dx}{\sqrt{1 - x^4}} = \frac{5}{2} \text{arc} \sin x^2 + C.$

4. $\displaystyle\int \frac{ax\,dx}{x^4 + e^4} = \frac{a}{2e^2} \text{arc} \tan \frac{x^2}{e^2} + C.$

5. $\displaystyle\int \frac{dx}{\sqrt{9 - 4x^2}} = \frac{1}{2} \sin^{-1} \frac{2x}{3} + C.$

6. $\displaystyle\int \frac{dx}{\sqrt{6 - x^2}} = \text{arc} \sin \frac{x}{\sqrt{6}} + C.$

7. $\displaystyle\int \frac{7\,ds}{\sqrt{3 - 5s^2}} = \frac{7}{\sqrt{5}} \text{arc} \sin \sqrt{\frac{5}{3}}\, s + C.$

8. $\int \frac{\cos \alpha d\alpha}{a^2 + \sin^2 \alpha} = \frac{1}{a} \arctan\left(\frac{\sin \alpha}{a}\right) + C.$

9. $\int \frac{e^t dt}{\sqrt{1 - e^{2t}}} = \arcsin e^t + C.$

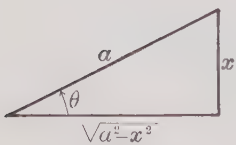
10. $\int \frac{dx}{x\sqrt{1 - \log^2 x}} = \arcsin(\log x) + C.$

11. $\int \frac{du}{\sqrt{a^2 - (u+b)^2}} = \arcsin\left(\frac{u+b}{a}\right) + C.$

12. $\int \frac{a dz}{(z-e)^2 + b^2} = \frac{a}{b} \arctan \frac{z-e}{b} + C.$

74. Trigonometric and other substitutions. A useful device for integration is afforded by simple transformation of the variable. The following examples illustrate this statement.

EXAMPLES



1. Work out $\int \frac{x^2 dx}{\sqrt{a^2 - x^2}}$.

Solution. Substitute

$$x = a \sin \theta.$$

$$\therefore dx = a \cos \theta \, d\theta,$$

$$\sqrt{a^2 - x^2} = \sqrt{a^2 - a^2 \sin^2 \theta} = a \cos \theta.$$

Hence $\int \frac{x^2 dx}{\sqrt{a^2 - x^2}} = \int \frac{a^2 \sin^2 \theta \cdot a \cos \theta \cdot d\theta}{a \cos \theta} = a^2 \int \sin^2 \theta \, d\theta$

$$= a^2 \left[\tfrac{1}{2}\theta - \tfrac{1}{4} \sin 2\theta\right].$$

(problem 16, p. 192)

From the figure, $\theta = \arcsin \frac{x}{a}$,

$$\sin 2\theta = 2 \sin \theta \cos \theta = 2 \frac{x}{a} \cdot \frac{\sqrt{a^2 - x^2}}{a}.$$

$$\therefore \int \frac{x^2 dx}{\sqrt{a^2 - x^2}} = \frac{a^2}{2} \arcsin \frac{x}{a} - \frac{1}{2} x \sqrt{a^2 - x^2} + C. \quad \textit{Ans.}$$

Integrals involving the radical $\sqrt{a^2 - x^2}$ may be worked in this way.

2. Work out $\int \frac{dx}{x^2\sqrt{a^2+x^2}}$.

Solution. Substitute $$x = \frac{1}{y}.$$

$\therefore\ dx = -\frac{dy}{y^2}$. Also $\sqrt{a^2+x^2} = \sqrt{a^2+\frac{1}{y^2}} = \frac{\sqrt{a^2y^2+1}}{y}$

Hence $$\int \frac{dx}{x^2\sqrt{a^2+x^2}} = \int \frac{-\frac{dy}{y^2}}{\frac{1}{y^2}\cdot\frac{\sqrt{a^2y^2+1}}{y}} = -\int \frac{ydy}{\sqrt{a^2y^2+1}}.$$

$$= -\frac{\sqrt{a^2y^2+1}}{a^2} \qquad \text{(by (4))}$$

$$= -\frac{\sqrt{a^2+x^2}}{a^2x} + C. \quad \textit{Ans.}$$

PROBLEMS

Work out:

1. $\int \frac{dx}{x^2\sqrt{a^2-x^2}} = -\frac{\sqrt{a^2-x^2}}{a^2x} + C.$ $\left[\text{Put } x = \frac{1}{y}.\right]$

2. $\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{1}{2}a^2 \text{ arc sin } \frac{x}{a} + C.$

3. $\int \frac{dx}{x\sqrt{x^2-a^2}} = -\frac{1}{a}\text{ arc sin } \frac{a}{x} + C.$ $\left[\text{Put } x = \frac{1}{y}.\right]$

4. $\int \frac{dz}{(1+z^2)^{\frac{3}{2}}} = \frac{z}{\sqrt{1+z^2}} + C.$ $\left[\text{Put } z = \frac{1}{y}.\right]$

5. $\int \frac{x^2\,dx}{(a^2-x^2)^{\frac{3}{2}}} = \frac{x}{\sqrt{a^2-x^2}} - \text{arc sin } \frac{x}{a} + C.$

6. $\int \frac{\sqrt{a^2-x^2}}{x^2}\,dx = -\frac{\sqrt{a^2-x^2}}{x} - \text{arc sin } \frac{x}{a} + C.$

CHAPTER XIV

CONSTANT OF INTEGRATION

75. Determination of the constant of integration by means of initial conditions. In order to determine the constant of integration, data must be given in addition to the differential expression to be integrated. Let us illustrate by means of examples.

EXAMPLES

1. Find a function whose first derivative is $3x^2 - 2x + 5$, and which shall have the value 12 when $x = 1$.

Solution. $(3x^2 - 2x + 5)\,dx$ is the differential expression to be integrated. Thus,

$$\int (3x^2 - 2x + 5)\,dx = x^3 - x^2 + 5x + C,$$

where C is the constant of integration. From the conditions of our problem this result must equal 12 when $x = 1$; that is,

$$12 = 1 - 1 + 5 + C, \text{ or } C = 7.$$

Hence $x^3 - x^2 - 5x + 7$ is the required function.

2. Find the laws governing the motion of a point which moves in a straight line with constant acceleration.

Solution. Since the acceleration $\left[= \dfrac{dv}{dt} \text{ from (6), Art. 67}\right]$ is constant, say f, we have

$$\frac{dv}{dt} = f,$$

or $$dv = fdt. \quad \text{Integrating,}$$

(A) $$v = ft + C.$$

To determine C, suppose that the *initial* velocity be v_0; that is, let

$$v = v_0 \text{ when } t = 0.$$

These values substituted in (A) give

$$v_0 = 0 + C, \text{ or } C = v_0.$$

Hence (A) becomes

$$(B) \qquad v = ft + v_0.$$

Since $v = \frac{ds}{dt}$ [Art. 67], we get from (B)

$$\frac{ds}{dt} = ft + v_0,$$

or

$$ds = ftdt + v_0dt. \quad \text{Integrating,}$$

$$(C) \qquad s = \tfrac{1}{2}ft^2 + v_0t + C.$$

To determine C, suppose that the *initial* space (= distance) be s_0; that is, let

$$s = s_0 \text{ when } t = 0.$$

These values substituted in (C) give

$$s_0 = 0 + 0 + C, \text{ or } C = s_0.$$

Hence (C) becomes

$$(D) \qquad s = \tfrac{1}{2}ft^2 + v_0t + s_0.$$

Formulas (B) and (D) are the required laws.

3. A certain magnitude z varies with the time according to the **Compound Interest Law**, that is, z and $\frac{dz}{dt}$ are proportional. Find z as a function of the time t.

Solution. By definition,

$$\frac{dz}{dt} = az,$$

where a is a constant factor of proportionality. Multiplying both sides by dt and dividing through by z gives

$$\frac{dz}{z} = adt.$$

Integrating,

(1) $$\log z = at + C.$$

To determine C, assume that the value of z when $t = 0$ is denoted by z_0. Substituting in (1), $C = \log z_0$.

We may now write (1) in the form

(2) $$\log z - \log z_0 = at, \text{ or } \log \frac{z}{z_0} = at.$$

Hence, changing to exponentials (Art. 33),

$$\frac{z}{z_0} = e^{at}, \text{ or } z = z_0 e^{at}. \quad \textit{Ans.}$$

PROBLEMS

Find the function whose first derivative is

1. $x - 3$, knowing that the function equals 9 when $x = 2$.

Ans. $\frac{x^2}{2} - 3x + 13.$

2. $3 + x - 5x^2$, knowing that the function equals -20 when $x = 6$.

Ans. $124 + 3x + \frac{x^2}{2} - \frac{5x^3}{3}.$

3. $(y^3 - b^2 y)\,dy$, knowing that the function equals 0 when $y = 2$.

Ans. $\frac{y^4}{4} - \frac{b^2 y^2}{2} + 2b^2 - 4.$

4. $\sin \alpha + \cos \alpha$, knowing that the function equals 2 when $\alpha = \frac{\pi}{2}$.

Ans. $\sin \alpha - \cos \alpha + 1.$

Find the equation of a curve such that the slope of the tangent at any point is

5. $3x - 2$. *Ans.* $y = \frac{3x^2}{2} - 2x + C.$

6. $x^2 + 5x$, the curve passing through the point (0, 3).

Ans. $y = \frac{x^3}{3} + \frac{5x^2}{2} + 3.$

7. $\frac{p}{y}$, the curve passing through the point (0, 0).

Ans. $y^2 = 2px$.

8. $\frac{b^2x}{a^2y}$, the curve passing through the point $(a, 0)$.

Ans. $b^2x^2 - a^2y^2 = a^2b^2$.

9. m, the curve making an intercept b on the axis of y.

Ans. $y = mx + b$.

Assuming that $v = v_0$ when $t = 0$, find the relation between v and t, knowing that the acceleration is

10. Zero. *Ans.* $v = v_0$.

11. Constant $= k$. *Ans.* $v = v_0 + kt$.

12. $a + bt$. *Ans.* $v = v_0 + at + \frac{bt^2}{2}$.

Assuming that $s = 0$ when $t = 0$, find the relation between s and t, knowing that the velocity is

13. Constant $(= v_0)$. *Ans.* $s = v_0 t$.

14. $m + nt$. *Ans.* $s = mt + \frac{nt^2}{2}$.

15. $3 + 2t - 3t^2$. *Ans.* $s = 3t + t^2 - t^3$.

16. The velocity of a body starting from rest is $5t^2$ ft. per second after t sec. (*a*) How far will it be from the point of starting in 3 sec.? (*b*) In what time will it pass over a distance of 360 ft. measured from the starting point?

Ans. (*a*) 45 ft.; (*b*) 6 sec.

17. A train starting from a station has after t hr. a speed of $t^3 - 21t^2 + 80t$ mi. per hour. Find (*a*) its distance from the station; (*b*) during what interval the train was moving backwards; (*c*) when the train repassed the station; (*d*) the distance the train had traveled when it passed the station the last time.

Ans. (*a*) $\frac{1}{4}t^4 - 7t^3 + 40t^2$ mi.;
(*b*) from 5th to 16th hr.;
(*c*) in 8 and 20 hr.;
(*d*) $4658\frac{1}{2}$ mi.

18. The equation giving the strength of the current i for the time t after the source of E.M.F. is removed, is (R and L being constants)

$$Ri = -L\frac{di}{dt}.$$

Find i, assuming that $I =$ current when $t = 0$. *Ans.* $i = Ie^{-\frac{Rt}{L}}$.

19. Find the current of discharge i from a condenser of capacity C in a circuit of resistance R, assuming the initial current to be I_0, having given the relation

$$\frac{di}{i} = \frac{dt}{CR},$$

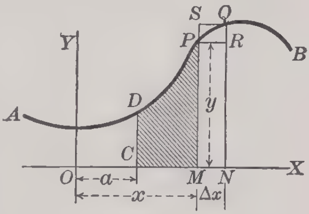
C and R being constants. *Ans.* $i = I_0 e^{\frac{t}{CR}}$.

CHAPTER XV

THE DEFINITE INTEGRAL

76. Differential of an area. Consider the curve AB whose equation is

$$y = \phi(x).$$



Let CD be a fixed and MP a variable ordinate, and let u be the measure of the area $CMPD$.* When x takes on a sufficiently small increment Δx, u takes on an increment Δu (= area $MNQP$). Completing the rectangles $MNRP$ and $MNQS$, we see that

$$\text{area } MNRP < \text{area } MNQP < \text{area } MNQS,$$

or

$$MP \cdot \Delta x < \Delta u < NQ \cdot \Delta x;$$

dividing by Δx,

$$MP < \frac{\Delta u}{\Delta x} < NQ.\dagger$$

Now let Δx approach zero as a limit; then since MP remains fixed and NQ approaches MP as a limit, we get

$$\frac{du}{dx} = y (= MP),$$

or, using differentials,

$$du = ydx.$$

* We may suppose this area to be generated by a variable ordinate starting out from CD and moving to the right; hence u will be a function of x, which vanishes when $x = a$.

† In this figure MP is less than NQ; if MP happens to be greater than NQ, simply reverse the inequality signs.

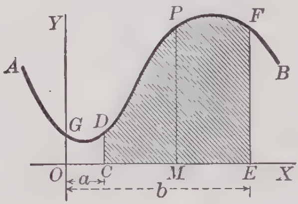
Theorem. *The differential of the area bounded by any curve, the axis of X, and a fixed and a variable ordinate is equal to the product of the variable ordinate and the differential of the corresponding abscissa.*

77. The definite integral. It follows from the theorem in the last section that if AB is the locus of

$$y = \phi(x),$$

then $$du = ydx, \text{ or}$$

(A) $$du = \phi(x)dx$$



is the differential of the area between the curve, the axis of x, and two ordinates. Integrating (A),

$$u = \int \phi(x)dx.$$

Let $\int \phi(x)dx$ be worked out and denote the result by $f(x) + C$.

(B) $$\therefore u = f(x) + C.$$

We may determine C, as in Chapter XIII, if we know the value of u for some value of x. If we agree to reckon the area from the axis of y, *i.e.* when

(C) $$x = a,\ u = \text{area } OCDG,$$

and when $$x = b,\ u = \text{area } OEFG, \text{ etc.,}$$
it follows that if

(D) $$x = 0, \text{ then } u = 0.$$

Substituting (D) in (B),

$$0 = f(0) + C, \text{ or } C = -f(0).$$

Hence from (B)

$$(E) \qquad u = f(x) - f(0),$$

giving the area from the axis of y to any ordinate (as MP).

To find the area between the ordinates CD and EF, we observe that

$$(F) \qquad \text{area } CEFD = \text{area } OEFG - \text{area } OCDG.$$

But, using (E),

$$\text{area } OCDG = f(a) - f(0),$$

$$\text{area } OEFG = f(b) - f(0).$$

Substituting in (F),

$$(G) \qquad \textbf{area } \boldsymbol{CEFD = f(b) - f(a).}$$

Theorem. *The difference of the values of* $\int ydx$ for $x = a$ *and* $x = b$ *gives the area bounded by the curve whose ordinate is* y, *the axis of* X, *and the ordinates corresponding to* $x = a$ *and* $x = b$.

This difference is represented by the symbol

$$(H) \qquad \int_a^b ydx, \text{ or } \int_a^b \phi(x)dx,$$

and is read "the integral from a to b of ydx." The operation is called *integration between limits*, a being the *lower* and b the *upper* limit.

Since (H) always has a *definite* value, it is called a *definite integral*. For, if

$$\int \phi(x)dx = f(x) + C,$$

then $\displaystyle \int_a^b \phi(x)dx = \Big[f(x) + C\Big]_a^b = \Big[f(b) + C\Big] - \Big[f(a) + C\Big],$

or

$$\int_a^b \phi(x)dx = f(b) - f(a),$$

the *constant of integration* having disappeared.

We may accordingly define the symbol

$$\int_a^b \phi(x)dx$$

as the numerical measure of the area bounded by the curve $y=\phi(x)$, *the axis of* X, *and the ordinates of the curve at* $x=a$, $x=b$. *This definition presupposes that these lines bound an area,* i.e. *the curve does not rise or fall to infinity, and both* a *and* b *are finite.*

78. Geometrical representation of an integral. In the last section we represented the definite integral as an area. This does not necessarily mean that every integral *is* an area, for the physical interpretation of the result depends on the nature of the quantities $\phi(x)$ and x. In the chapter on functions, the variable x was chosen to represent magnitudes of various kinds — time, length, etc. The corresponding function might be a volume, or any other physical magnitude. The definite integral (H) is then represented in *numerical value* by the area in the *graph* of $\phi(x)$, but its actual physical significance might be something quite different. For example, if we turn to equation (6), p. 168, and represent the acceleration as a function of the time by $a(t)$, then, multiplying through by dt and integrating, we obtain the indefinite integral $v=\int a(t)dt$. The difference of two values of this integral, that is, the definite integral, is clearly a change in velocity, and hence is a velocity. Further illustrations occur in later sections.

79. Calculation of a definite integral. The process may be summarized as follows:

First step. *Find the indefinite integral of the given differential expression.*

Second step. *Substitute in this indefinite integral first the upper limit and then the lower limit for the variable, and subtract the last result from the first.*

It is not necessary to bring in the constant of integration, since it always disappears in subtracting.

EXAMPLES

1. Find $\int_1^4 x^2dx.$

Solution. $\int_1^4 x^2dx = \left[\frac{x^3}{3}\right]_1^4 = \frac{64}{3} - \frac{1}{3} = 21.$ *Ans.*

2. Find $\int_0^\pi \sin x dx.$

Solution. $\int_0^\pi \sin x dx = \left[-\cos x\right]_0^\pi = [-(-1)] - [-1] = 2.$ *Ans.*

3. Find $\int_0^a \frac{dx}{a^2+x^2}.$

Solution. $\int_0^a \frac{dx}{a^2+x^2} = \left[\frac{1}{a}\arctan\frac{x}{a}\right]_0^a = \frac{1}{a}\arctan 1 - \frac{1}{a}\arctan 0$

$= \frac{\pi}{4a} - 0 = \frac{\pi}{4a}.$ *Ans.*

PROBLEMS

1. $\int_2^3 6x^2dx = 38.$

2. $\int_0^a (a^2x - x^3)dx = \frac{a^4}{4}.$

3. $\int_1^4 \frac{dx}{x^{\frac{3}{2}}} = 1.$

4. $\int_1^e \frac{dx}{x} = 1.$

5. $\int_0^1 (x^2 - 2x + 2)(x-1)dx = -\frac{3}{4}.$

6. $\int_0^1 \frac{dx}{\sqrt{3-2x}} = \sqrt{3} - 1.$

7. $\int_0^2 \frac{x^3dx}{x+1} = \frac{8}{3} - \log 3.$

8. $\int_0^{\frac{1}{\sqrt{3}}} \frac{dx}{\sqrt{2-3x^2}} = \frac{\pi}{4\sqrt{3}}.$

9. $\int_2^3 \frac{3xdx}{2\sqrt[4]{x^2-4}} = \sqrt[4]{125}.$

10. $\int_0^1 \frac{dy}{y^2-y+1} = \frac{2\pi}{3\sqrt{3}}.$

11. $\int_2^3 \frac{tdt}{1+t^2} = \frac{\log 2}{2}.$

12. $\int_0^{+\infty} \frac{dx}{a^2+x^2} = \frac{\pi}{2a}.$

13. $\int_0^{\frac{\pi}{2}} \sin\phi d\phi = 1.$

14. $\int_0^{+\infty} e^{-x}dx = 1.$

15. $\int_0^{2r} \frac{\sqrt{2r}}{\sqrt{x}}dx = 4r.$

16. $\int_0^5 \left(\frac{3}{5}\sqrt{t} - \frac{3}{25}t^2\right) dt = 2\sqrt{5} - 5.$

17. $\int_0^r \frac{r\,dx}{\sqrt{r^2 - x^2}} = \frac{\pi r}{2}.$

18. $\int_0^{2r} \frac{2\sqrt{2\,r}\,dy}{\sqrt{2\,r - y}} = 8\,r.$

19. $\int_{-b}^{b} \frac{\pi}{a^4}(y^2 - b^2)^4\,dy = \frac{256\,\pi\,b^9}{315\,a^4}.$

20. $2\,a\int_0^{\pi} (2+2\cos\theta)^{\frac{1}{2}}\,d\theta = 8\,a.$

21. $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \tan\alpha\,d\alpha = 0.$

CHAPTER XVI

INTEGRATION A PROCESS OF SUMMATION

80. Introduction. Thus far we have defined integration as the *inverse of differentiation.* In a great many of the applications of the Integral Calculus, however, it is preferable to define integration as a *process of summation.* In fact the Integral Calculus was invented in the attempt to calculate the area bounded by curves by supposing the given area to be divided up into an "infinite number of infinitesimal parts called *elements*, the sum of all these elements being the area required." Historically the integral sign is merely the long S, used by early writers to indicate "sum."

This new definition, as amplified in the next section, is of fundamental importance, and it is essential that the student should thoroughly understand what is meant in order to be able to apply the Integral Calculus to practical problems.

81. The fundamental theorem of the Integral Calculus. If $\phi(x)$ is the derivative of $f(x)$, then it has been shown in § 77, p. 203, that the value of the definite integral

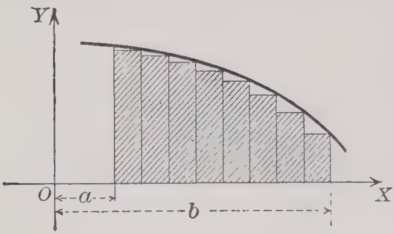
$$(A) \qquad \int_a^b \phi(x)dx = f(b) - f(a)$$

gives the area bounded by the curve $y = \phi(x)$, the x-axis, and ordinates erected at $x = a$ and $x = b$.

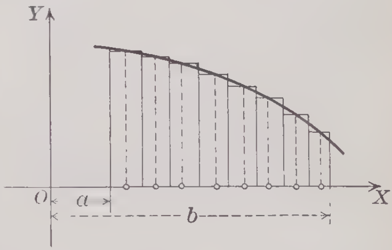
Let us now make the following construction in connection with this area.

Divide the interval from $x = a$ to $x = b$ into any number n of equal subintervals, erect ordinates at the points of division, and complete rectangles by drawing horizontal lines through the

extremities of the ordinates, as in the figure. It is clear that the *sum* of the areas of these n rectangles (the shaded area) is an approximate value for the area in question. It is further evident that the *limit* of the sum of the areas of these rectangles, when their number n is indefinitely increased, will *equal* the area under the curve.



Let us now carry through the following more general construction. Divide the interval into n subintervals, *not necessarily equal*, and erect ordinates at the points of division. Choose a point within each subinterval in any manner, erect ordinates at these points, and through their extremities draw horizontal lines to form rectangles as in the figure. Then, as before, the sum of the areas of these n rectangles equals approximately the area under the curve, and the *limit of this sum* as n increases without limit and each subinterval approaches zero as a limit is precisely the area under the curve.



These considerations show that the definite integral (A) may be regarded as *the limit of a sum*. Let us now formulate this result.

1. Denote the lengths of the successive subintervals by

$$\Delta x_1, \Delta x_2, \Delta x_3, \cdots, \Delta x_n.$$

2. Denote the abscissas of the points chosen in the subintervals by

$$x_1, x_2, x_3, \cdots, x_n.$$

Then the ordinates of the curve at these points are

$$\phi(x_1),\ \phi(x_2),\ \phi(x_3),\ \cdots,\ \phi(x_n).$$

3. The areas of the successive rectangles are obviously

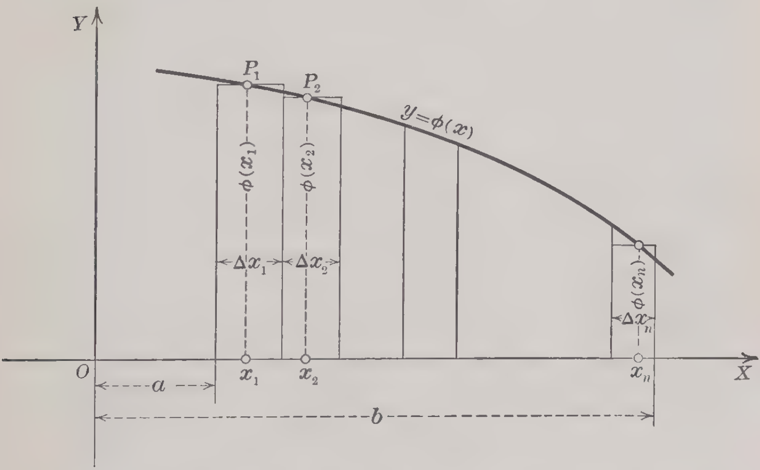
$$\phi(x_1)\Delta x_1,\ \phi(x_2)\Delta x_2,\ \cdots,\ \phi(x_n)\Delta x_n.$$

4. The area under the curve is therefore equal to

$$\underset{n=\infty}{\text{limit}}\left[\phi(x_1)\Delta x_1 + \phi(x_2)\Delta x_2 + \cdots + \phi(x_n)\Delta x_n\right].$$

The discussion gives the equation

$$(B)\quad \int_a^b \phi(x)dx = \underset{n=\infty}{\text{limit}}\left[\phi(x_1)\Delta x_1 + \phi(x_2)\Delta x_2 + \cdots + \phi(x_n)\Delta x_n\right].$$



The equation (B) has been established by making use of the notion of area. The area under discussion is *bounded*, that is, it has a closed perimeter consisting of the curve $y=\phi(x)$, the x-axis, and the lines $x=a$, $x=b$. It is therefore tacitly assumed that the function $\phi(x)$ is *finite* for all values of x from a to b inclusive. That is, the curve $y=\phi(x)$ does not run off to infinity between $x=a$ and $x=b$; otherwise expressed, the

curve has no "breaks" in it. The student may refer to Art. 32 for examples of curves which have "breaks" in them; that is, have vertical asymptotes. The property of the function $\phi(x)$ here described, that is, of remaining finite between $x = a$ and $x = b$, and yielding a continuous graph, is expressed by the word "*continuous.*"

Intuition has aided us in establishing the result (B). Let us now regard (B) *simply as a theorem in analysis*, which may be stated thus:

Fundamental theorem of the Integral Calculus. *Let $\phi(x)$ be continuous for the interval $x = a$ to $x = b$. Let this interval be divided into n subintervals whose lengths are $\Delta x_1, \Delta x_2, \cdots, \Delta x_n$, and let points be chosen, one in each subinterval, their abscissas being $x_1, x_2, \cdots, x_n$ respectively. Consider the sum*

$$(C) \qquad \phi(x_1)\,\Delta x_1 + \phi(x_2)\,\Delta x_2 + \cdots + \phi(x_n)\,\Delta x_n.$$

Then the limiting value of this sum when n increases without limit and each subinterval approaches zero as a limit equals the value of the definite integral $\int_a^b \phi(x)\,dx$.

The importance in the application of this theorem is this: *We are able to calculate by integration any magnitude which is the limit of a sum of the form (C).*

The difficulty which arises in actual practice is that of determining the function $\phi(x)$. No general rule applicable to all cases can be given for overcoming this difficulty. The student should study the examples worked out in the following pages in order to obtain practice.

We may observe that the terms of the sum (C) are of the form

$$(D) \qquad \phi(x)\,\Delta x, \text{ or also } \phi(x)\,dx \text{ (since } dx = \Delta x),$$

which may be called the *general term*, or also an *element of the integral.*

The following directions will be found useful in applying the Fundamental Theorem.

First step. Divide the required magnitude into similar parts such that it is clear that the desired result will be found by summing up these parts and passing to the limit.

Second step. Choose a suitable variable such that the magnitude of each part can be expressed in the form (D).

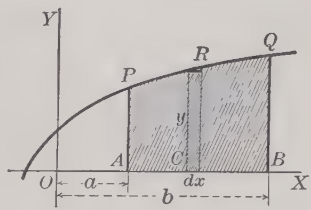
Third step. Apply the Fundamental Theorem and integrate.

82. Areas of plane curves. As already explained, the area between a curve, the axis of X, and the ordinates $x = a$ and $x = b$ is given by the formula

$$(A) \qquad \textbf{\textit{area}} = \int_a^b y\,dx,$$

the value of y in terms of x being substituted from the equation of the curve.

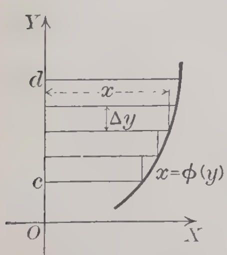
Equation (A) is readily memorized by observing that the integrand ydx represents the area of a rectangle CR of base dx and altitude y.



The application of the Fundamental Theorem to the calculation of the area bounded by the curve

$$(1) \qquad x = \phi(y),$$

the y-axis and abscissas at $y = c$ and $y = d$, is immediate.



First step. Construct the n rectangles as in the figure. The area is clearly the limit of the sum of these rectangles as their number increases indefinitely.

Second step. Call any one of the altitudes Δy. The base of any one of the rectangles is x. Hence the *general expression* for the area of the rectangles is $x\,\Delta y$ or $\phi(y)\,\Delta y$, from (1). This is an *element* of the required integral.

Third step. Applying the Fundamental Theorem gives the formula

$$\text{area} = \int_c^d \phi(y)\,dy.$$

The result may be stated: The area between a curve, the Y-axis, and abscissas at $y = c$ and $y = d$ is given by

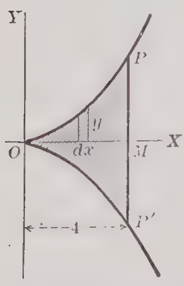
$$(\boldsymbol{B}) \qquad \boldsymbol{area} = \int_c^d \boldsymbol{x\,dy},$$

the value of x in terms of y being found from the equation of the curve.

EXAMPLES

1. Find the area included between the semicubical parabola $y^2 = x^3$ and the line $x = 4$.

Solution. Let us first find the area OMP, half of the required area OPP'. For the upper branch of the curve $y = \sqrt{x^3}$, and summing up the rectangles between the limits $x = 0$ and $x = 4$, we get, by substituting in (A),



$$\text{area } OMP = \int_0^4 y\,dx = \int_0^4 x^{\frac{3}{2}}dx = \tfrac{64}{5} = 12\tfrac{4}{5}.$$

Hence area $OPP' = 2 \cdot \frac{64}{5} = 25\frac{3}{5}$.

If the unit of length is one inch, the area of OPP is $25\frac{3}{5}$ sq. in.

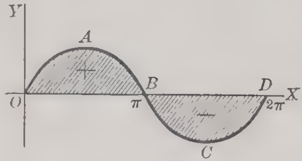
Note. For the lower branch $y = -x^{\frac{3}{2}}$; hence

$$\text{area } OMP' = \int_0^4 (-x^{\frac{3}{2}})\,dx = -12 \cdot \tfrac{4}{5}.$$

This area lies below the axis of x and has a negative sign because the ordinates are negative.

In finding the area OMP above, *the result was positive* because the ordinates were positive, *the area lying above the axis of x.*

The above result, $25\frac{3}{5}$, was the total area regardless of sign. As we shall illustrate in the next example, it is important to note the sign of the area when the curve crosses the axis of X within the limits of integration.



2. Find the area of one arch of the sine curve $y = \sin x$.

Solution. Placing $y = 0$ and solving for x, we find

$$x = 0,\ \pi,\ 2\pi, \text{ etc.}$$

Substituting in (A), p. 211,

$$\text{area } OAB = \int_a^b y\,dx = \int_0^\pi \sin x\,dx = 2.$$

Also, $$\text{area } BCD = \int_a^b y\,dx = \int_\pi^{2\pi} \sin x\,dx = -2,$$

and $$\text{area } OABCD = \int_a^b y\,dx = \int_0^{2\pi} \sin x\,dx = 0.$$

This last result takes into account the signs of the two separate areas composing the whole. The total area regardless of these signs equals 4.

3. Find the area included between the parabola $x^2 = 4\,ay$ and the witch

$$y = \frac{8\,a^3}{x^2 + 4\,a^2}.$$

Solution. To determine the limits of integration, we solve the equations simultaneously to find where the curves intersect. The coördinates of A are found to be $(-2\,a, a)$, and of B $(2\,a, a)$.

It is seen from the figure (p. 214) that

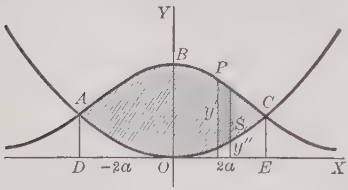
$$\text{area } AOCB = \text{area } DECBA - \text{area } DECOA.$$

But $$\text{area } DECBA = 2 \times \text{area } OECB = 2\int_0^{2a} \frac{8\,a^3 dx}{x^2 + 4\,a^2} = 2\,\pi a^2,$$

and $$\text{area } DECOA = 2 \times \text{area } OEC = 2\int_0^{2a} \frac{x^2}{4\,a}dx = \frac{4\,a^2}{3}.$$

Hence $$\text{area } AOCB = 2\,\pi a^2 - \frac{4\,a^2}{3} = 2\,a^2\left(\pi - \frac{2}{3}\right). \quad \textit{Ans.}$$

Another method is to consider the strip PS as an element of the area. If y' is the ordinate corresponding to the witch, and y'' to the parabola, the differential expression for the area of the strip PS equals $(y'-y'')dx$. Substituting the values of y' and y'' in terms of x from the given equations, we get

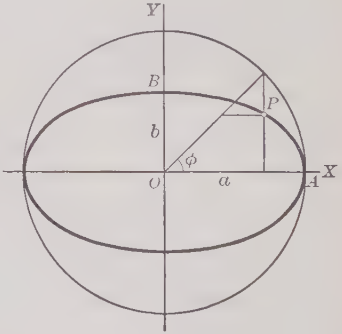


$$\text{area } AOCB = 2 \times \text{area } OCB$$
$$= 2\int_0^{2a}(y'-y'')\,dx$$
$$= 2\int_0^{2a}\left(\frac{8a^3}{x^2+4a^2}-\frac{x^2}{4a}\right)dx$$
$$= 2a^2(\pi - \tfrac{2}{3}).$$

4. Find the area of the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$.

Solution. To find the area of the quadrant OAB, the limits are $x=0$, $x=a$; and

$$y=\frac{b}{a}\sqrt{a^2-x^2}.$$



Hence, substituting in (A), p. 211,

$$\text{area } OAB = \frac{b}{a}\int_0^a (a^2-x^2)^{\frac{1}{2}}dx$$
$$=\left[\frac{bx}{2a}(a^2-x^2)^{\frac{1}{2}}+\frac{ab}{2}\text{arc}\sin\frac{x}{a}\right]_0^a$$
$$=\frac{\pi ab}{4}. \qquad \text{(problem 2, § 74)}$$

Therefore the entire area of the ellipse equals πab.

We may also calculate the integral giving area OAB as follows.

Let $x=a\cos\phi$ (see figure and § 74). Then $\sqrt{a^2-x^2}=a\sin\phi$, $dx=-a\sin\phi\, d\phi$.

Changing the limits: — when $x=0$, $\phi=\frac{1}{2}\pi$; when $x=a$, $\phi=0$.

$$\therefore \frac{b}{a}\int_0^a (a^2-x^2)^{\frac{1}{2}}dx = -ab\int_{\frac{1}{2}\pi}^0 \sin^2\phi\, d\phi$$
$$=-ab\left[\frac{\phi}{2}-\frac{1}{4}\sin 2\phi\right]_{\frac{1}{2}\pi}^0=\frac{\pi ab}{4}, \text{ as before.}$$

PROBLEMS

1. Find the area bounded by the line $y = 5x$, the x-axis, and $x = 2$. *Ans.* 10.

2. Find the area bounded by the parabola $y^2 = 4x$, the axis of X, and the lines $x = 4$ and $x = 9$. *Ans.* $25\frac{1}{3}$.

3. Find the area bounded by the parabola $y^2 = 4x$, the axis of Y, and the lines $y = 4$ and $y = 6$. *Ans.* $12\frac{2}{3}$.

4. Find the area of the circle $x^2 + y^2 = r^2$. *Ans.* πr^2.

5. Find the area between the equilateral hyperbola $xy = a^2$, the axis of X, and the ordinates $x = a$, $x = 2a$. *Ans.* $a^2 \log 2$.

6. Find the area between the curve $y = 4 - x^2$ and the axis of X. *Ans.* $10\frac{2}{3}$.

7. Find the area intercepted between the coördinate axes and the parabola $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$. *Ans.* $\frac{1}{6}a^2$.

8. Find the area bounded by the semicubical parabola $y^2 = x^3$, the axis of Y, and the line $y = 4$. *Ans.* $\frac{3}{5}\sqrt[3]{1024}$.

9. Find the area between the catenary $y = \frac{a}{2}[e^{\frac{x}{a}} + e^{-\frac{x}{a}}]$, the axis of Y, and the line $x = a$. *Ans.* $\frac{a^2}{2e}[e^2 - 1]$.

10. Find the area between the witch $y = \frac{8a^3}{x^2 + 4a^2}$ and the axis of X, its asymptote. *Ans.* $4\pi a^2$.

11. Find the area included between the two parabolas $y^2 = 2px$ and $x^2 = 2py$. *Ans.* $\frac{4}{3}p^2$.

12. Find the total area included between the curve $y = x^3$ and the line $y = 2x$. *Ans.* 2.

13. Find by integration the area of the triangle bounded by the axis of Y and the lines $2x + y + 8 = 0$, $y = -4$. *Ans.* 4

14. Find the area bounded by each of the following curves and the X-axis.

(a) $y = 9 - x^2$. *Ans.* 36.

(b) $y = 4 - x^2$. *Ans.* $10\frac{2}{3}$.

(c) $y = 3 - 2x - x^2$. *Ans.* $10\frac{2}{3}$.

(d) $y = 5 - 4x - x^2$.

(e) $y = \sin x + \cos x$ (one arch). *Ans.* $2\sqrt{2}$.

(f) $y = x\sqrt{4 - x^2}$. *Ans.* $2\frac{2}{3}$.

(g) $y = x\sqrt{9 - x^2}$.

15. Find the area bounded as described.

(a) $y = \dfrac{4}{1 + x}$, ordinates at $x = 0$, $x = 3$. *Ans.* $8 \log_e 2$.

(b) $y = \dfrac{4}{(1 + x)^2}$ ordinates at $x = 0$, $x = 3$. *Ans.* 3.

(c) $y = \tan x$, ordinates at $x = 0$, $x = \dfrac{\pi}{4}$. *Ans.* $\frac{1}{2} \log_e 2$.

(d) $y = \sqrt{4 - 2x}$, ordinates at $x = 0$, $x = 2$. *Ans.* $2\frac{2}{3}$.

(e) $y = \sqrt{10 - 3x}$, ordinates at $x = 1$, $x = 3$.

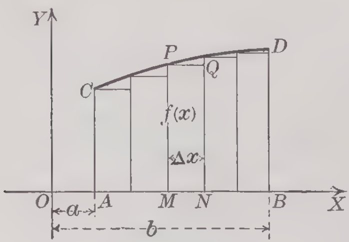
Ans. $\frac{2}{9}[7^{\frac{3}{2}} - 1]$.

(f) $x = 9 - y^2$, and the y-axis. *Ans.* 36.

(g) $x = \sin^2 y$, and the y-axis. *Ans.* $\frac{1}{2}\pi$.

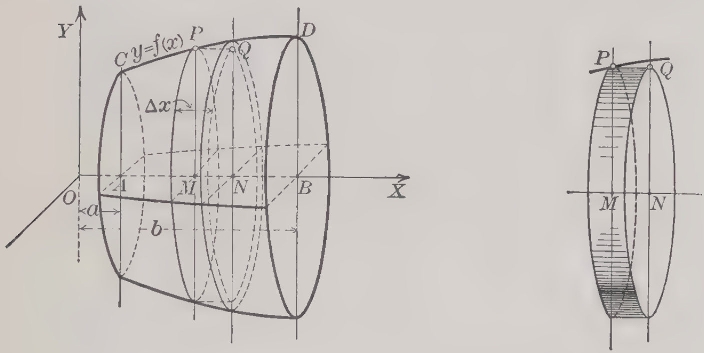
83. Volumes of solids of revolutions. Let V denote the volume of the solid generated by revolving the plane surface $ACDB$ about the axis of X, the equation of the plane curve CPD being

$$(1) \qquad y = f(x).$$



Construct rectangles as in the figure. When the area is revolved each rectangle generates a cylinder of revolution. The required volume is clearly equal to the *limit* of the sum of the volumes of these cylinders.

Consider any one of the cylinders, say the one generated by the rectangle $MNPQ$. Let $MN = \Delta x$, $MP = y = f(x)$. Then



the volume of the cylinder generated is $\pi y^2 \Delta x = \pi f(x)^2 \Delta x$, and hence the expression for an *element* of the required integral is

(2) $$\pi y^2 \Delta x, \text{ or } \pi f(x)^2 \Delta x.$$

The Fundamental Theorem now gives the formula

(*A*) $$\boldsymbol{V = \pi \int_a^b y^2 dx,}$$

where the value of y in terms of x must be substituted from the equation (1) of the given curve.

This formula is easily recalled if we consider a slice or disk of the solid between two planes perpendicular to the axis of revolution as an element of the volume, and regard it as a cylinder of infinitesimal altitude dx and with a base of area πy^2, — hence of volume $\pi y^2 dx$. Summing up all such slices (elements), we get the entire volume. Similarly, when OY is the axis of revolution, we use the formula

(*B*) $$\boldsymbol{V = \pi \int_c^d x^2 dy,}$$

where the value of x in terms of y must be substituted from the equation of the given curve.

EXAMPLE

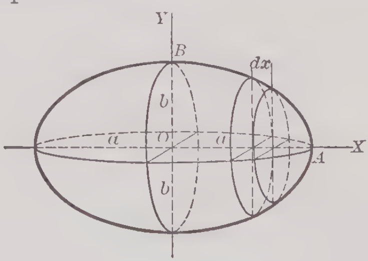
1. Find the volume generated by revolving the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ about the axis of X.

Solution. Since $y^2=\frac{b^2}{a^2}(a^2-x^2)$, and the required volume is twice the volume generated by OAB, we get, substituting in (A),

$$\frac{V}{2}=\pi\int_0^a y^2dx=\pi\int_0^a\frac{b^2}{a^2}(a^2-x^2)\,dx=\frac{2\,\pi ab^2}{3}.$$

$$\therefore\ V=\frac{4\,\pi ab^2}{3}.$$

To verify this result, let $b=a$. Then $V=\frac{4\,\pi a^3}{3}$, the volume of a sphere, which is only a special case of the ellipsoid. When the ellipse is revolved about its major axis the solid generated is called a prolate spheroid; when about its minor axis, an oblate spheroid.



PROBLEMS

1. Find the volume of the sphere generated by revolving the circle $x^2+y^2=r^2$ about a diameter. *Ans.* $\frac{4\,\pi r^3}{3}$.

2. Find by integration the volume of the right cone generated by revolving the line joining the origin to the point (a, b) about the axis of X. Verify your result geometrically.

Ans. $\frac{\pi ab^2}{3}$.

3. Find the volume of the cone generated by revolving the line of problem 2 about the axis of Y. Verify your result geometrically. *Ans.* $\frac{\pi a^2 b}{3}$.

4. Find the volume of the paraboloid of revolution generated by revolving the arc of the parabola $y^2 = 4ax$ between the origin and the point (x_1, y_1) about its axis.

Ans. $2\pi a x_1^2 = \frac{\pi y_1^2 x_1}{2}$; *i.e.* one half of the volume of the circumscribing cylinder.

5. Find the volume generated by revolving the arc in problem 4 about the axis of Y.

Ans. $\frac{\pi y_1^5}{80 a^2} = \frac{1}{5}\pi x_1^2 y_1$; *i.e.* one fifth of the cylinder of altitude y_1 and radius of base x_1.

6. Find by integration the volume of the cone generated by revolving about OX that part of the line $4x - 5y + 3 = 0$ which is intercepted between the coördinate axes. *Ans.* $\frac{9\pi}{100}$.

7. Find the volume of the torus (ring) generated by revolving the circle $x^2 + (y - b)^2 = a^2$ about OX. *Ans.* $2\pi^2 a^2 b$.

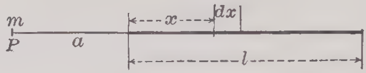
8. Find the volume generated by revolving one arch of the sine curve $y = \sin x$ about the axis of X. *Ans.* $\frac{\pi^2}{2}$.

9. Find the volumes of the solids obtained by revolving around XX' each of the areas of problem 14, p. 216, and problem 15 (a)–(e), p. 216.

84. Miscellaneous applications of the Integral Calculus. The following examples illustrate further the principle of summation. The student should study carefully the method of subdividing the magnitude to be calculated into elements, the expression for an element ultimately becoming an element of the required integral ((D), p. 210).

EXAMPLES

1. Determine the amount of attraction exerted by a thin, straight, homogeneous rod of uniform thickness, of length l, and of mass M, upon a material point P of mass m situated at a distance of a from one end of the rod in its line of direction.



Solution. Suppose the rod to be divided into equal infinitesimal portions (elements) of length dx.

$$\frac{M}{l} = \text{mass of a unit length of rod;}$$

hence $$\frac{M}{l}\,dx = \text{mass of any element.}$$

Newton's Law for measuring the attraction between any two masses is

$$\text{force of attraction} = \frac{\text{product of masses}}{(\text{distance between them})^2};$$

therefore the force of attraction between the particle at P and an element of the rod is

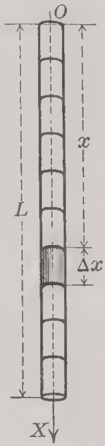
$$\frac{\frac{M}{l}\,m\,dx}{(x+a)^2},$$

which is then an *element of the force of attraction required.* The total attraction between the particle at P and the rod being the sum of all such elements between $x=0$ and $x=l$, we have

$$\text{force of attraction} = \int_0^l \frac{\frac{M}{l}\,m\,dx}{(x+a)^2}$$

$$= \frac{Mm}{l}\int_0^l \frac{dx}{(x+a)^2} = -\frac{Mm}{l(a+l)}. \quad \textit{Ans.}$$

2. A heavy cable of length L ft. hangs vertical. Its weight per foot is w lb. Find the work done in foot-pounds when the cable is hauled up.



Solution. Draw the X-axis from the upper extremity downward. Consider the cable as made up of small pieces each of length Δx. Then the weight of each of these pieces is $w\,\Delta x$. Consider any one of these pieces whose distance from the upper end of the cable is x. Then the work done in hauling up this piece is the product of the weight and the height it is raised; that is,

work done in raising one piece $= x \cdot w\,\Delta x.$

Hence the expression

$$wx\,\Delta x$$

is an element of the required integral.

The total work done in hauling in the cable is therefore given by the integral

$$\text{work} = \int_0^L xw\,dx = w\int_0^L x\,dx = \frac{wL^2}{2}\text{ ft. lb.}$$

The weight of the cable ($= W$) is of course wL.

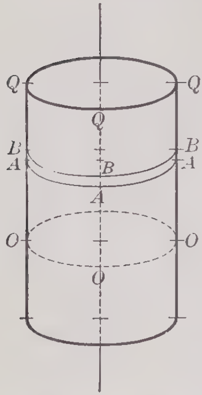
$$\therefore \text{work done} = W\tfrac{1}{2}L;$$

that is, *the work done is equal to that of raising a weight equal to that of the cable a height equal to one half the length of the cable.*

3. A perfect gas in a cylinder expands from the volume v_0 to the volume v_1, the temperature remaining constant. Find the work done.

Solution. Let the original level (when the volume is v_0) be the circle OOO, and the final level (volume $= v_1$) the circle QQQ. Consider now the expansion from the level AA to BB, the height AB being small.

Since the temperature remains constant, the pressure and the volume during the expansion satisfy Boyle's Law,



(1) $$pv = \text{constant} = k.$$

In this formula, p is the pressure per unit area.

Now the work done when the gas expands from the level AA to BB is the product of the total pressure and the height AB.

Let the cross section of the cylinder have the area δ. Then the total pressure equals $p\delta$. The volume between the levels AA and BB, which we denote by Δv, is given by $\Delta v = \delta \cdot AB$. $\therefore AB = \dfrac{\Delta v}{\delta}$. Hence the element of work done, that is,

(2) *the work done when the volume increases by* $\Delta v = p\delta \cdot AB$
$$= p\,\Delta v.$$

This must be expressed in terms of v. By (1), $p = \dfrac{k}{v}$, $\therefore$ element of work done $= k\dfrac{dv}{v}$. The Fundamental Theorem applies at once, and gives

$$\textit{work done} = \int_{v_0}^{v_1} k\frac{dv}{v} = k\int_{v_0}^{v_1}\frac{dv}{v} = k\log\frac{v_1}{v_0}. \quad \textit{Ans.}$$

PROBLEMS

1. A quantity of steam expands so as to satisfy the law $pv^{1.13} = C$. When $v = 1$ cu. ft., $p = 8000$ lb. per square foot. Find the work done in expansion from $v = 3$ to $v = 10$.

Ans. 7750 ft. lb.

2. Find the work done in the expansion of a quantity of steam from 2 cu. ft. at a pressure of 2 T. per square foot to 8 cu. ft. The law of expansion is $pv^{0.9} = C$. *Ans.* 5.97 ft. T.

3. A point moves along a line so that the relation between the velocity and time is one of the following. Find the distance it will move in the interval of time indicated.

(a) $v = \frac{1}{3}t$; $t = 0$ to $t = 2$ sec. *Ans.* $\frac{2}{3}$.

(b) $v = 16 - 32t$; $t = 0$ to $t = 1$ sec. *Ans.* 0.

(c) $v = 3\cos t$; $t = 0$ to $t = \frac{\pi}{2}$. *Ans.* 3.

(d) $v = \frac{2}{1 + 6t}$; $t = 0$ to $t = 4$. *Ans.* 1.07.

4. A sluiceway is closed by a rectangular gate 10 ft. wide by 12 ft. deep. The top of the gate is level with the water. Find the pressure against the gate, assuming that 1 cu. ft. of water weighs 62 lb. *Ans.* 44,640 lb.

5. Find the amount of the attraction in Ex. 1, Art. 84, if the material point P is directly over the center of the rod and at a distance a from it.

Ans. $\frac{Mm}{al}\theta$, where θ is the angle subtended by the rod for an observer at P.

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